6.2 Diagonalizing a Matrix

Note Title 1/18/2008

1. Factor these two matrices into $A = S\Lambda S^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

(a) Find λ's : $\begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0$

$\therefore (\lambda - 3)(\lambda - 1) = 0 \Rightarrow \lambda = 1, 3$

Find eigenvectors :

$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\therefore 3x_2 = x_2 \Rightarrow x_2 = 0$

$x_1 + 2x_2 = x_1 + x_2$

$\Rightarrow x_1 = x_1$.

$\therefore \lambda = 1$; $c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1 + 2x_2 = 3x_1$, $x_1 = x_2$

$3x_2 = 3x_2$

$\therefore \lambda = 3$: $c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\therefore S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$\therefore A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

(b) λ's : $\begin{vmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = 0$, $(\lambda-2)(\lambda-1)-2=0$

$\lambda^2 - 3\lambda = 0$

$$\therefore \lambda = 0, 3$$

Eigenvectors:

$\lambda = 0$: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_1 + x_2 = 0$

$x_1 = -x_2$

$$\therefore c\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda = 3$: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$ $x_1 + x_2 = 3x_1$

$2x_1 + 2x_2 = 3x_2$

$\therefore x_2 = 2x_1$, $\therefore c\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\therefore S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \quad S^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$$

2. If $A = S\Lambda S^{-1}$ then $A^3 = (\quad)(\quad)(\quad)$ and $A^{-1} = (\quad)(\quad)(\quad)$.

(a) $A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda S^{-1}S\Lambda S^{-1}$

$= S\Lambda^2 S^{-1}$

$\therefore A^3 = \underline{S\Lambda^3 S^{-1}}$

(b) $A = S\Lambda S^{-1} \Rightarrow A^{-1} = (S\Lambda S^{-1})^{-1} = \underline{S\Lambda^{-1}S^{-1}}$,

if Λ is invertible.

3. If A has $\lambda_1 = 2$ with eigenvector $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and x's.

$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 0 & 5 \end{bmatrix} = \underline{\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}}$

5. True or false: If the columns of S (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) S is invertible (d) S is diagonalizable.

(a) A may not be invertible if one of the λ's is 0. $\therefore$ False.

(b) This is 6D: $\Lambda = S^{-1}AS$, or

$AS = [\lambda_1 x_1 \cdots \lambda_n x_n] = S\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} = S\Lambda$.
$\therefore$ True.

(c) True

(d) False: just because a matrix is invertible doesn't mean there are eigenvectors.

i.e., Suppose $\det A \neq 0$. $\det(A - \lambda I) = 0$ may not have any real roots.
$\therefore$ over the field of reals, there may be no eigenvectors, $\therefore$ can't diagonalize.

8. Write down the most general matrix that has eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$.

Consider $\begin{bmatrix}a & c\\ b & d\end{bmatrix}$. $\therefore \begin{bmatrix}a & c\\ b & d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \lambda_1\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\lambda_1\\ \lambda_1\end{bmatrix}$

$\therefore \begin{bmatrix}a & c\\ b & d\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \lambda_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}\lambda_2\\ -\lambda_2\end{bmatrix}$

$\therefore \quad a + c = \lambda_1 \qquad a - c = \lambda_2$

$\qquad b + d = \lambda_1 \qquad b - d = -\lambda_2$

$2a = \lambda_1 + \lambda_2 \qquad a = \dfrac{\lambda_1 + \lambda_2}{2}$

$2b = \lambda_1 - \lambda_2 \qquad b = \dfrac{\lambda_1 - \lambda_2}{2}$

$2c = \lambda_1 - \lambda_2 \qquad c = \dfrac{\lambda_1 - \lambda_2}{2}$

$2d = \lambda_1 + \lambda_2 \qquad d = \dfrac{\lambda_1 + \lambda_2}{2}$

$\therefore \quad a = d, \; b = c, \; \text{so} \quad \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

18. The matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable because the rank of $A - 3I$ is _____. Change one entry to make A diagonalizable. Which entries could you change?

Note $\det(A - \lambda I) = (3 - \lambda)^2$, so look at $A - 3I$, its rank is $\underline{1}$, so dim nullspace is 1, so can't have two independent eigenvectors. $\therefore$ not diagonalizable.

Changing a_{11}, a_{21}, or a_{22} can make $\det(A - \lambda I)$ have two different roots.

Changing a_{12} won't change determinant.

20. (Recommended) Find Λ and S to diagonalize A in Problem 19. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of this limiting matrix you see the _____.

(a) $A = \begin{bmatrix} .6 & .4 \\ .4 & .6 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} .6-\lambda & .4 \\ .4 & .6-\lambda \end{vmatrix}$$

$$= \lambda^2 - 1.2\lambda + 0.36 - 0.16$$

$$= \lambda^2 - 1.2\lambda + .20$$

$$= (\lambda - 1.0)(\lambda - 0.2)$$

$\therefore \lambda = 1, 0.2 \quad \therefore \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}$

Eigenvectors: $\lambda = 1$ $\begin{bmatrix} -.4 & .4 \\ .4 & -.4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left.\begin{aligned} -.4x + .4y &= 0 \\ .4x - .4y &= 0 \end{aligned}\right\} \quad x = y$$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 0.2$ $\begin{bmatrix} .4 & .4 \\ .4 & .4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left.\begin{aligned} .4x + .4y &= 0 \\ .4x + .4y &= 0 \end{aligned}\right\}$ $x = -y$

$\therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $S^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\therefore A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

(b) $\lim_{K\to\infty} \Lambda^K = \lim_{K\to\infty} \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\lim_{K\to\infty} S\Lambda^K S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

This is the <u>steady state</u>.

24. Suppose that $A = S\Lambda S^{-1}$. Take determinants to prove that $\det A = \lambda_1\lambda_2\cdots\lambda_n =$ product of λ's. This quick proof only works when A is _____.

$$\det A = \det(S)\det(\Lambda)\det(S^{-1})$$
$$= \det(S)\det(S^{-1})\det(\Lambda)$$
$$= \det(SS^{-1})\det(\Lambda)$$
$$= \det(I)\det(\Lambda)$$
$$= \det(\Lambda) = \lambda_1\lambda_2\cdots\lambda_n$$

Only works when A is diagonalizable.

30. (Recommended) Suppose $Ax = \lambda x$. If $\lambda = 0$ then x is in the nullspace. If $\lambda \neq 0$ then x is in the column space. Those spaces have dimensions $(n-r)+r=n$. So why doesn't every square matrix have n linearly independent eigenvectors?

x (solutions to $Ax = 0 \cdot x$) will give $n-r$ independent eigenvectors.

for $\lambda \neq 0$, solution to $Ax = \lambda x$ may not give r independent eigenvectors.