6.4 Symmetric Matrices

Note Title 2/15/2017

1. **1** Write A as $M+N$, symmetric matrix plus skew-symmetric matrix:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = M + N \qquad (M^T = M, N^T = -N).$$

For any square matrix, $M = \frac{A+A^T}{2}$ and $N =$ _____ add up to A.

$$M+N = \begin{bmatrix} 1 & a & c \\ a & 3 & e \\ c & e & 5 \end{bmatrix} + \begin{bmatrix} 0 & -b & -d \\ b & 0 & -f \\ d & f & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix}$$

$\therefore$ $\left.\begin{array}{l} a+b=4 \\ a-b=2 \end{array}\right\}$ $\begin{array}{l} 2a=6 \\ a=3,\ b=1 \end{array}$ $\left.\begin{array}{l} c+d=8 \\ c-d=4 \end{array}\right\}$ $\begin{array}{l} 2c=12,\ c=6 \\ d=2 \end{array}$

$\left.\begin{array}{l} e+f=6 \\ e-f=0 \end{array}\right\}$ $\begin{array}{l} 2e=6,\ e=3 \\ f=3 \end{array}$

$\therefore$ $M = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix}$ $N = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$

2. **2** If C is symmetric prove that A^TCA is also symmetric. (Transpose it.) When A is 6 by 3, what are the shapes of C and A^TCA?

$(A^TCA)^T = A^TC^T(A^T)^T = A^TCA.$

A^T is 3×6 $\therefore$ $(3\times 6)(\quad)(6\times 3) = A^TCA$

$\Rightarrow C$ is 6×6, and $A^T C A$ is 3×3

3. **3** Find the eigenvalues and the unit eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda & \lambda \\ 1 & 0 & -\lambda \end{vmatrix} = \begin{vmatrix} 0 & 1 & -\lambda^2+\lambda+1 \\ 0 & -\lambda & \lambda \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$= \lambda + \lambda(-\lambda^2+\lambda+1) = -\lambda^3 + \lambda^2 + 2\lambda = 0$$

$\therefore \lambda = 0$ and $\lambda^2 - \lambda - 2 = 0$, $(\lambda-2)(\lambda+1) = 0$

or $\lambda = 2, -1$.

$\therefore \underline{\lambda = 0, -1, 2}$

$\lambda = 0$: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\therefore x = (0, a, -a)$

$\therefore \|x\| = 1 \Rightarrow \sqrt{2a^2} = 1$, $a = \pm \frac{1}{\sqrt{2}}$

$$\therefore x = \pm \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$\lambda = -1$: $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore x_1 = -x_3, \quad x_2 = x_3$

$\therefore (a, -a, -a) = x$

$\|x\| = 1 \Rightarrow \sqrt{3a^2} = 1, \quad a = \pm \frac{1}{\sqrt{3}}$

$\therefore x = \pm \frac{1}{\sqrt{3}} (1, -1, -1)$

$\lambda = 2$: $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to$

$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_2 = x_3 \; , \; -x_1 + x_2 + x_2 = 0 \Rightarrow 2x_2 = x_1$

$\therefore x = (2a, a, a)$

$||x|| = 1 \Rightarrow \sqrt{6a^2} = 1, \; a = \pm \frac{1}{\sqrt{6}}$

$\therefore x = \pm \frac{1}{\sqrt{6}} (2, 1, 1)$

4. **4** Find an orthogonal matrix Q that diagonalizes $A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$.

$A = A^T \; \therefore$ real eigenvalues, perpendicular eigenvectors

$$\therefore \det\left(\begin{bmatrix} -2-\lambda & 6 \\ 6 & 7-\lambda \end{bmatrix}\right) = (\lambda-7)(\lambda+2) - 36$$
$$= \lambda^2 - 5\lambda - 50$$
$$= (\lambda - 10)(\lambda + 5) = 0, \; \lambda = 10, -5$$

$$\lambda = 10: \begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{or } \vec{x} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda = -5: \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \text{ or } \vec{y} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\therefore A \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} = \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix}^T$$

$$\therefore A = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

using $Q = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

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6. **6** Find *all* orthogonal matrices that diagonalize $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.

$$\det\left(\begin{bmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{bmatrix}\right) = (\lambda-9)(\lambda-16) - 144 = \lambda^2 - 25\lambda = 0$$

$$\Rightarrow \lambda = 0, 25$$

$\lambda = 0$; $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 & 12 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, or $\vec{x} = \pm \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix} = \pm \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}$

$\lambda = 25$: $\begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -16 & 12 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \pm \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, or $\vec{y} = \pm \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \pm \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$

$\therefore Q = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$ or $\begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$

or $\begin{bmatrix} 0.8 & -0.6 \\ -0.6 & -0.8 \end{bmatrix}$ or $\begin{bmatrix} -0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$

or switch columns on the above matrices

8. **8** If $A^3 = 0$ then the eigenvalues of A must be _____. Give an example that has $A \neq 0$. But if A is symmetric, diagonalize it to prove that A must be zero.

(a) If $A\vec{x} = \lambda\vec{x}$, $\vec{x} \neq \vec{0}$, then $A^3\vec{x} = \lambda^3\vec{x}$

$A^3 = 0 \Rightarrow A^3\vec{x} = 0 \Rightarrow \lambda^3\vec{x} = 0 \Rightarrow \lambda^3 = 0$

(b) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\therefore \lambda = 0$ for all eigenvalues

(c) $A = A^T \Rightarrow A = Q\Lambda Q^T \Rightarrow A^3 = Q\Lambda^3 Q^T$

But $\Lambda^3 = \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$.

$\therefore A = Q\Lambda Q^T = Q(0)Q^T = 0$. $\therefore \underline{A = 0}$

9. **9** If $\lambda = a + ib$ is an eigenvalue of a real matrix A, then its conjugate $\bar{\lambda} = a - ib$ is also an eigenvalue. (If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then also $A\bar{\boldsymbol{x}} = \bar{\lambda}\bar{\boldsymbol{x}}$.) Prove that every real 3 by 3 matrix has a real eigenvalue.

If A is 3×3, it has 3 eigenvalues as solutions to $\det(A - \lambda I) = 0$. Two of the eigenvalues are conjugates: $\lambda_2 = \bar{\lambda}_1$.

But $\lambda_1 + \lambda_2 + \lambda_3 = \text{trace} = \text{real}$

$= (\lambda_1 + \bar{\lambda}_1) + \lambda_3 = \text{real} + \lambda_3 = \text{real}$,

$\therefore \lambda_3 = \text{real}$.

10. **10** Here is a quick "proof" that the eigenvalues of all real matrices are real:

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{gives} \quad \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = \lambda\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \quad \text{so} \quad \lambda = \frac{\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}} \quad \text{is real.}$$

Find the flaw in this reasoning—a hidden assumption that is not justified.

$\vec{x}$ may not be real, so $\vec{x}^T\vec{x}$ may be complex,

and $\vec{x}^T A \vec{x}$ may be complex.

11. **11** Write A and B in the form $\lambda_1 x_1 x_1^{\mathrm{T}} + \lambda_2 x_2 x_2^{\mathrm{T}}$ of the spectral theorem $Q\Lambda Q^T$:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \qquad (\text{keep } \|x_1\| = \|x_2\| = 1).$$

A: $\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) = \lambda^2 - 6\lambda + 8 = 0,$
$(\lambda-4)(\lambda-2) = 0, \ \lambda = 4, 2$

$\lambda = 2$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 4$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \vec{y} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore A = 2\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} + 4\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

or $A = 2\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1/2 & 1/2 \end{bmatrix} + 4\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1/2 & 1/2 \end{bmatrix}$

B: From #6 above, $\lambda = 0, 25$

$$\lambda = 0: \quad \vec{x} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$$

$$\lambda = 25: \quad \vec{y} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

$$\therefore B = 0 \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix} \begin{bmatrix} 4/5 & -3/5 \end{bmatrix} + 25 \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 \end{bmatrix}$$

$$\text{or } B = 0 \begin{bmatrix} 16/25 & -12/25 \\ -12/25 & 9/25 \end{bmatrix} + 25 \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$$

12. **12** Every 2 by 2 symmetric matrix is $\lambda_1 \boldsymbol{x}_1 \boldsymbol{x}_1^{\mathrm{T}} + \lambda_2 \boldsymbol{x}_2 \boldsymbol{x}_2^{\mathrm{T}} = \lambda_1 P_1 + \lambda_2 P_2$. Explain $P_1 + P_2 = \boldsymbol{x}_1 \boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{x}_2 \boldsymbol{x}_2^{\mathrm{T}} = I$ from columns times rows, $P_1 P_2 = \boldsymbol{x}_1 \boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{x}_2 \boldsymbol{x}_2^{\mathrm{T}} = 0$ and from row times column.

A 2×2 symmetric matrix has 2 orthogonal eigenvectors (spectral theorem).

Call them $\vec{x}$ and $\vec{y}$, and assume $\|\vec{x}\| = 1$, $\|\vec{y}\| = 1$.

$\therefore [\vec{x} \ \ \vec{y}]$ is orthogonal and

$$\vec{x} \cdot \vec{x}^T = P_1 \quad , \quad \vec{y} \cdot \vec{y}^T = P_2$$

$\therefore$

$$P_1 + P_2 = \vec{x} \cdot \vec{x}^T + \vec{y} \cdot \vec{y}^T = [\vec{x}][\vec{x}^T] + [\vec{y}][\vec{y}^T]$$

$$= [\vec{x} \ \ \vec{y}] \begin{bmatrix} \vec{x}^T \\ \vec{y}^T \end{bmatrix} = [\vec{x} \ \ \vec{y}][\vec{x} \ \ \vec{y}]^T$$

$$= Q Q^T = I$$

$$P_1 P_2 = \left([\vec{x}][\vec{x}^T]\right)\left([\vec{y}][\vec{y}^T]\right)$$

$$= [\vec{x}]\left([\vec{x}^T][\vec{y}]\right)[\vec{y}^T]$$

$$= [\vec{x}]\left([\vec{x}^T \cdot \vec{y}]\right)[\vec{y}^T]$$

$$= \underset{2\times 1}{[\vec{x}]}\ \underset{1\times 1}{([0])}\ \underset{1\times 2}{[\vec{y}^T]} = \underset{2\times 2}{[0]}$$

18. **18** ***Another proof that eigenvectors are perpendicular when $A = A^T$.*** Suppose $Ax = \lambda x$ and $Ay = 0y$ and $\lambda \neq 0$. Then y is in the nullspace and x is in the column space. They are perpendicular because ______. Go carefully—why are these subspaces orthogonal? If the second eigenvalue is a nonzero number β, apply this argument to $A - \beta I$. The eigenvalue moves to zero and the eigenvectors stay the same—so they are perpendicular.

(a) $\vec{x} \perp \vec{y}$ because the nullspace is perpendicular to the row space = column space for symmetric matrices.

(b) $(A-\beta I)x = Ax - \beta x = \lambda x - \beta x = (\lambda-\beta)x$

$\therefore \vec{x}$ is in column space of $A-\beta I$ (assuming $\lambda \neq \beta$).

$(A-\beta I)y = Ay - \beta y = \beta y - \beta y = 0$

$\therefore \vec{y}$ is in nullspace of $A-\beta I$.

But $(A-\beta I)^T = A^T - \beta I^T = A - \beta I$.

$\therefore$ column space = row space for $A-\beta I$.

$\therefore \vec{y} \perp$ row space $\Rightarrow \vec{y} \perp$ column space

$\Rightarrow \vec{y} \perp \vec{x}$

21. **21** True or false. Give a reason or a counterexample.

(a) A matrix with real eigenvalues and eigenvectors is symmetric.

(b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

(c) The inverse of a symmetric matrix is symmetric.

(d) The eigenvector matrix S of a symmetric matrix is symmetric.

(a) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\therefore (\lambda-1)^2 - 0 \Rightarrow \lambda = 1, 1$

Eigenvectors: $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ just $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) If the eigenvectors are orthogonal, then S = matrix of eigenvectors is invertible.
$\therefore AS = S\Lambda$, so $A = S\Lambda S^{-1}$.
But can create Q from S by making eigenvectors unit vectors. $\therefore A = Q\Lambda Q^{-1}$.
Since Q is orthogonal, $Q^T = Q^{-1}$.
$\therefore A = Q\Lambda Q^T$

$\therefore A^T = (Q\Lambda Q^T)^T = Q\Lambda^T Q^T = Q\Lambda Q^T$
since $\Lambda = \Lambda^T$

$\therefore A$ is symmetric

(c) True, since $(A^T)^{-1} = (A^{-1})^T$

$\therefore (A^{-1})^T = (A^T)^{-1} = (A)^{-1}$

(d) False. Just look at Q of Example 2 on p. 321 of text.

22. **22** A *normal matrix* has $A^{\mathrm{T}}A = AA^{\mathrm{T}}$; it has orthogonal eigenvectors. Why is every skew-symmetric matrix normal? Why is every orthogonal matrix normal? When is $\begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}$ normal?

(a) If A is skew-symmetric, then $A^T = -A$.

$$\therefore\ A^TA = (-A)A = -A^2 \quad \text{and}$$
$$AA^T = A(-A) = -A^2$$

$$\therefore\ A^TA = AA^T$$

(b) If A is orthogonal, then $A^TA = I$, so A^T is the inverse of A, and $\therefore AA^T = I$.
$\therefore\ AA^T = A^TA = I$.

(c) $$\begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}\begin{bmatrix} a & -1 \\ 1 & d \end{bmatrix} = AA^T = \begin{bmatrix} a^2+1 & -a+d \\ -a+d & 1+d^2 \end{bmatrix}$$

$$\begin{bmatrix} a & -1 \\ 1 & d \end{bmatrix}\begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix} = A^TA = \begin{bmatrix} a^2+1 & a-d \\ a-d & 1+d^2 \end{bmatrix}$$

$$\therefore\ -a+d = a-d, \text{ or } 2d = 2a,\ \underline{d=a}$$

Note: A^TA and AA^T are symmetric, and $\therefore$ have real eigenvalues and orthogonal eigenvectors.

23. **23** (A paradox for instructors) If $AA^T = A^TA$ then A and A^T share the same eigenvectors (true). A and A^T always share the same eigenvalues. Find the flaw in this conclusion: They must have the same S and Λ. Therefore A equals A^T.

Look at $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \therefore A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \therefore A \neq A^T$

But $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^TA$

Both A and A^T have:

$\det(A - \lambda I) = \det(A^T - \lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda = i, -i$

But for A: $\lambda = i \Rightarrow A - \lambda I = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow x = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\lambda = -i \Rightarrow A - \lambda I = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \rightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow x = \begin{bmatrix} -1 \\ i \end{bmatrix}$

$\therefore S\Lambda = \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

For A^T: $\lambda = i \Rightarrow A^T - \lambda I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \to \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow x = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\lambda = -i \Rightarrow A^T - \lambda I = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \to \begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow x = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\therefore S\Lambda = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$\therefore A$ has $S = \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix}$

A^T has $S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$

$\therefore$ share same eigenvectors, but not in same relationship with the shared eigenvalues

24. **24** (Recommended) Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B: LU, QR, $S\Lambda S^{-1}$, $Q\Lambda Q^{\mathrm{T}}$?

A: Column vectors are independent and orthogonal.

$\therefore$ A is invertible and orthogonal

$A^T = A$, so A is diagonalizable

The sum of the column entries is 1 for each column. $\therefore$ A is a Markov matrix.

Since $A^TA = I$, then A is a permutation matrix, a permutation is orthogonal, so $P^{-1} = P^T$. $\therefore P^TP = I$.

Since A is invertible, $\lambda = 0$ is not an eigenvalue. $\therefore$ A cannot "project" vectors onto a subspace of the column vectors.

A invertible $\Rightarrow$ can do $A = LU$
also, column vectors are independent so

can do Gram-Schmidt, so can do
$\underline{A = QR}$

A symmetric $\Rightarrow$ can do $\underline{Q \Lambda Q^T}$
which is a special case of $\underline{S \Lambda S^{-1}}$

B: Column vectors dependent so not invertible,
not orthogonal.

$B^T B \neq I$, so B not a permutation

Since B is singular, $\lambda = 0$ is an
eigenvalue, so B can project vectors
onto a subspace of column vectors
$\therefore$ a projection matrix

Since $B = B^T$, B is diagonalizable

Since column entries add to 1, B is Markov

Can always do $\underline{B = LU}$, although U
will not be invertible. (zero in a pivot).

Since columns of B are dependent, can't do Gram-Schmidt. $\therefore$ no $B = QR$

Since $B = B^T$, can do $B = Q \Lambda Q^T$,

which is a special case of $S \Lambda S^{-1}$