

6.4 Symmetric Matrices

Note Title

2/15/2017

- 1 Write A as $M + N$, symmetric matrix plus skew-symmetric matrix:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = M + N \quad (M^T = M, N^T = -N).$$

For any square matrix, $M = \frac{A+A^T}{2}$ and $N = \frac{A-A^T}{2}$ add up to A .

$$M + N = \begin{bmatrix} 1 & a & c \\ a & 3 & e \\ c & e & 5 \end{bmatrix} + \begin{bmatrix} 0 & -b & -d \\ b & 0 & -f \\ d & f & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix}$$

$$\therefore \left. \begin{array}{l} a+b=4 \\ a-b=2 \end{array} \right\} \begin{array}{l} 2a=6 \\ a=3, b=1 \end{array} \quad \left. \begin{array}{l} c+d=8 \\ c-d=4 \end{array} \right\} \begin{array}{l} 2c=12, c=6 \\ d=2 \end{array}$$

$$\left. \begin{array}{l} e+f=6 \\ e-f=0 \end{array} \right\} \begin{array}{l} 2e=6, e=3 \\ f=3 \end{array} \quad N = \frac{A-A^T}{2} \therefore M+N = \frac{A+A^T+A-A^T}{2} = A$$

$$\therefore M = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} \quad N = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$$

2. 2 If C is symmetric prove that $A^T C A$ is also symmetric. (Transpose it.) When A is 6 by 3, what are the shapes of C and $A^T C A$?

$$(A^T C A)^T = A^T C^T (A^T)^T = A^T C A.$$

$$A^T \text{ is } 3 \times 6 \therefore (3 \times 6)(\quad)(6 \times 3) = A^T C A$$

$\Rightarrow C$ is 6×6 , and $A^T C A$ is 3×3

3. 3 Find the eigenvalues and the unit eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda & \lambda \\ 1 & 0 & -\lambda \end{vmatrix} = \begin{vmatrix} 0 & 1 & -\lambda^2 + \lambda + 1 \\ 0 & -\lambda & \lambda \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$= \lambda + \lambda(-\lambda^2 + \lambda + 1) = -\lambda^3 + \lambda^2 + 2\lambda = 0$$

$$\therefore \lambda = 0 \text{ and } \lambda^2 - \lambda - 2 = 0, (\lambda - 2)(\lambda + 1) = 0 \\ \text{or } \lambda = 2, -1.$$

$$\therefore \underline{\lambda = 0, -1, 2}$$

$$\lambda = 0: \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \therefore x = (0, a, -a)$$

$$\therefore \|x\| = 1 \Rightarrow \sqrt{2a^2} = 1, a = \pm \frac{1}{\sqrt{2}}$$

$$\therefore \underline{\underline{x = \pm \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)}}$$

$$\lambda = -1: \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = -x_3, \quad x_2 = x_3$$

$$\therefore (a, -a, -a) = x$$

$$\|x\| = 1 \Rightarrow \sqrt{3a^2} = 1, \quad a = \pm \frac{1}{\sqrt{3}}$$

$$\therefore x = \pm \frac{1}{\sqrt{3}} (1, -1, -1)$$

$$\lambda = 2: \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_2 = x_3, \quad -x_1 + x_2 + x_2 = 0 \Rightarrow 2x_2 = x_1$$

$$\therefore x = (2a, a, a)$$

$$\|x\| = 1 \Rightarrow \sqrt{6a^2} = 1, \quad a = \pm \frac{1}{\sqrt{6}}$$

$$\therefore x = \pm \frac{1}{\sqrt{6}} (2, 1, 1)$$

4. 4 Find an orthogonal matrix Q that diagonalizes $A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$.

$A = A^T$ \therefore real eigenvalues, perpendicular eigenvectors

$$\begin{aligned} \therefore \det \begin{pmatrix} -2-\lambda & 6 \\ 6 & 7-\lambda \end{pmatrix} &= (\lambda-7)(\lambda+2) - 36 \\ &= \lambda^2 - 5\lambda - 50 \\ &= (\lambda-10)(\lambda+5) = 0, \quad \lambda = 10, -5 \end{aligned}$$

$$\lambda = 10: \begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{or } \vec{x} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda = -5: \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \text{ or } \vec{y} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\therefore A \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} = \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix}^T$$

$$\therefore A = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

using $Q = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

6. 6 Find all orthogonal matrices that diagonalize $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.

$$\det \left(\begin{bmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{bmatrix} \right) = (\lambda-9)(\lambda-16) - 144 = \lambda^2 - 25\lambda = 0$$
$$\Rightarrow \lambda = 0, 25$$

$$\lambda = 0; \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \text{ or } \vec{x} = \pm \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix} = \pm \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}$$

$$\lambda = 25: \begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -16 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \pm \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \text{ or } \vec{y} = \pm \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \pm \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$$

$$\therefore Q = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix} \text{ or } \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 0.8 & -0.6 \\ -0.6 & -0.8 \end{bmatrix} \text{ or } \begin{bmatrix} -0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

or switch columns on the above matrices

8.

8 If $A^3 = 0$ then the eigenvalues of A must be _____. Give an example that has $A \neq 0$. But if A is symmetric, diagonalize it to prove that A must be zero.

$$(a) \text{ If } A\vec{x} = \lambda\vec{x}, \vec{x} \neq \vec{0}, \text{ then } A^3\vec{x} = \lambda^3\vec{x}$$

$$A^3 = 0 \Rightarrow A^3\vec{x} = 0 \Rightarrow \lambda^3\vec{x} = 0 \Rightarrow \lambda^3 = 0$$

(b) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\therefore \lambda = 0$ for all eigenvalues

$$(c) A = A^T \Rightarrow A = Q \Lambda Q^T \Rightarrow A^3 = Q \Lambda^3 Q^T$$

$$\text{But } \Lambda^3 = \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

$$\therefore A = Q \Lambda Q^T = Q(0)Q^T = 0. \quad \therefore \underline{\underline{A = 0}}$$

9. 9 If $\lambda = a + ib$ is an eigenvalue of a real matrix A , then its conjugate $\bar{\lambda} = a - ib$ is also an eigenvalue. (If $Ax = \lambda x$ then also $A\bar{x} = \bar{\lambda}\bar{x}$.) Prove that every real 3 by 3 matrix has a real eigenvalue.

If A is 3×3 , it has 3 eigenvalues as solutions to $\det(A - \lambda I) = 0$. Two of the eigenvalues are conjugates: $\lambda_2 = \bar{\lambda}_1$.

$$\text{But } \lambda_1 + \lambda_2 + \lambda_3 = \text{trace} = \text{real}$$

$$= (\lambda_1 + \bar{\lambda}_1) + \lambda_3 = \text{real} + \lambda_3 = \text{real},$$

$$\therefore \lambda_3 = \text{real}.$$

10. 10 Here is a quick "proof" that the eigenvalues of all real matrices are real:

$$Ax = \lambda x \text{ gives } x^T Ax = \lambda x^T x \text{ so } \lambda = \frac{x^T Ax}{x^T x} \text{ is real.}$$

Find the flaw in this reasoning—a hidden assumption that is not justified.

\vec{x} may not be real, so $\vec{x}^T \vec{x}$ may be complex,
and $\vec{x}^T A \vec{x}$ may be complex.

11. 11 Write A and B in the form $\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T$ of the spectral theorem $Q \Lambda Q^T$:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \quad (\text{keep } \|\mathbf{x}_1\| = \|\mathbf{x}_2\| = 1).$$

$$A: \det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 8 = 0, \\ (\lambda - 4)(\lambda - 2) = 0, \lambda = 4, 2$$

$$\lambda = 2: \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 4: \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{y} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore A = 2 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} + 4 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\text{or } A = 2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 4 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

B : From # 6 above, $\lambda = 0, 25$

$$\lambda = 0: \vec{x} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$$

$$\lambda = 25: \vec{y} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

$$\therefore B = 0 \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix} \begin{bmatrix} 4/5 & -3/5 \end{bmatrix} + 25 \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 \end{bmatrix}$$

$$\text{or } B = 0 \begin{bmatrix} 16/25 & -12/25 \\ -12/25 & 9/25 \end{bmatrix} + 25 \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$$

12. 12 Every 2 by 2 symmetric matrix is $\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T = \lambda_1 P_1 + \lambda_2 P_2$. Explain $P_1 + P_2 = \mathbf{x}_1 \mathbf{x}_1^T + \mathbf{x}_2 \mathbf{x}_2^T = I$ from columns times rows, $P_1 P_2 = \mathbf{x}_1 \mathbf{x}_1^T + \mathbf{x}_2 \mathbf{x}_2^T = 0$ and from row times column.

A 2×2 symmetric matrix has 2 orthogonal eigenvectors (spectral theorem).

Call them \vec{x} and \vec{y} , and assume $\|\vec{x}\| = 1$, $\|\vec{y}\| = 1$.

$\therefore [\vec{x} \ \vec{y}]$ is orthogonal and

$$\vec{x} \cdot \vec{x}^T = \rho_1, \quad \vec{y} \cdot \vec{y}^T = \rho_2$$

$$\rho_1 + \rho_2 = \vec{x} \cdot \vec{x}^T + \vec{y} \cdot \vec{y}^T = [\vec{x}] [\vec{x}^T] + [\vec{y}] [\vec{y}^T]$$

$$= [\vec{x} \ \vec{y}] \begin{bmatrix} \vec{x}^T \\ \vec{y}^T \end{bmatrix} = [\vec{x} \ \vec{y}] [\vec{x} \ \vec{y}]^T$$

$$= Q Q^T = I$$

$$\rho_1 \rho_2 = ([\vec{x}] [\vec{x}^T]) ([\vec{y}] [\vec{y}^T])$$

$$= [\vec{x}] ([\vec{x}^T] [\vec{y}]) [\vec{y}^T]$$

$$= [\vec{x}] ([\vec{x}^T \cdot \vec{y}]) [\vec{y}^T]$$

$$= \underset{2 \times 1}{[\vec{x}]} \underset{1 \times 1}{([0])} \underset{1 \times 2}{[\vec{y}^T]} = \underset{2 \times 2}{[0]}$$

18.

18 Another proof that eigenvectors are perpendicular when $A = A^T$. Suppose $Ax = \lambda x$ and $Ay = 0y$ and $\lambda \neq 0$. Then y is in the nullspace and x is in the column space. They are perpendicular because _____. Go carefully—why are these subspaces orthogonal? If the second eigenvalue is a nonzero number β , apply this argument to $A - \beta I$. The eigenvalue moves to zero and the eigenvectors stay the same—so they are perpendicular.

(a) $\vec{x} \perp \vec{y}$ because the nullspace is

perpendicular to The row space = column space for symmetric matrices.

$$(5) (A - \beta I)x = Ax - \beta x = \lambda x - \beta x = (\lambda - \beta)x$$

$\therefore \vec{x}$ is in column space of $A - \beta I$ (assuming $\lambda \neq \beta$).

$$(A - \beta I)y = Ay - \beta y = \beta y - \beta y = 0$$

$\therefore \vec{y}$ is in nullspace of $A - \beta I$.

$$\text{But } (A - \beta I)^T = A^T - \beta I^T = A - \beta I.$$

\therefore column space = row space for $A - \beta I$.

$\therefore \vec{y} \perp \text{row space} \Rightarrow \vec{y} \perp \text{column space}$

$$\Rightarrow \vec{y} \perp \vec{x}$$

21.

21 True or false. Give a reason or a counterexample.

- (a) A matrix with real eigenvalues and eigenvectors is symmetric.
- (b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.
- (c) The inverse of a symmetric matrix is symmetric.
- (d) The eigenvector matrix S of a symmetric matrix is symmetric.

(a) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \therefore (\lambda - 1)^2 - 0 \Rightarrow \lambda = 1, 1$

Eigenvectors: $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ just $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) If the eigenvectors are orthogonal, then $S =$ matrix of eigenvectors is invertible.

$\therefore AS = S\Lambda$, so $A = SAS^{-1}$.

But can create Q from S by making eigenvectors unit vectors. $\therefore A = Q\Lambda Q^{-1}$.

Since Q is orthogonal, $Q^T = Q^{-1}$.

$\therefore A = Q\Lambda Q^T$

$\therefore A^T = (Q\Lambda Q^T)^T = Q\Lambda^T Q^T = Q\Lambda Q^T$
since $\Lambda = \Lambda^T$

$\therefore A$ is symmetric

(c) True, since $(A^T)^{-1} = (A^{-1})^T$

$\therefore (A^{-1})^T = (A^T)^{-1} = (A)^{-1}$

(d) False. Just look at Q of Example 2 on p. 321 of text.

22.

22 A normal matrix has $A^T A = A A^T$; it has orthogonal eigenvectors. Why is every skew-symmetric matrix normal? Why is every orthogonal matrix normal? When is $\begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}$ normal?

(a) If A is skew-symmetric, then $A^T = -A$.

$$\therefore A^T A = (-A)A = -A^2 \quad \text{and}$$

$$A A^T = A(-A) = -A^2$$

$$\therefore A^T A = A A^T$$

(b) If A is orthogonal, then $A^T A = I$, so A^T is the inverse of A , and $\therefore A A^T = I$.
 $\therefore A A^T = A^T A = I$.

$$(c) \begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix} \begin{bmatrix} a & -1 \\ 1 & d \end{bmatrix} = A A^T = \begin{bmatrix} a^2+1 & -a+d \\ -a+d & 1+d^2 \end{bmatrix}$$

$$\begin{bmatrix} a & -1 \\ 1 & d \end{bmatrix} \begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix} = A^T A = \begin{bmatrix} a^2+1 & a-d \\ a-d & 1+d^2 \end{bmatrix}$$

$$\therefore -a+d = a-d, \quad \text{or } 2d = 2a, \quad \underline{d = a}$$

Note: $A^T A$ and $A A^T$ are symmetric, and \therefore have real eigenvalues and orthogonal eigenvectors.

23. 23 (A paradox for instructors) If $AA^T = A^T A$ then A and A^T share the same eigenvectors (true). A and A^T always share the same eigenvalues. Find the flaw in this conclusion: They must have the same S and Λ . Therefore A equals A^T .

$$\text{Look at } A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \therefore A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \therefore A \neq A^T$$

$$\text{But } AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^T A$$

Both A and A^T have:

$$\det(A - \lambda I) = \det(A^T - \lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda = i, -i$$

$$\begin{aligned} \text{But for } A: \lambda = i \Rightarrow A - \lambda I &= \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \\ \Rightarrow x &= \begin{bmatrix} 1 \\ i \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \lambda = -i \Rightarrow A - \lambda I &= \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \rightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \\ \Rightarrow x &= \begin{bmatrix} -1 \\ i \end{bmatrix} \end{aligned}$$

$$\therefore S\Lambda = \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\text{For } A^T: \lambda = i \Rightarrow A^T - \lambda I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix}$$
$$\Rightarrow x = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\lambda = -i \Rightarrow A^T - \lambda I = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \rightarrow \begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}$$
$$\Rightarrow x = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\therefore S A = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\therefore A \text{ has } S = \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix}$$

$$A^T \text{ has } S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

\therefore share same eigenvectors, but not in same relationship with the shared eigenvalues

24.

- 24 (Recommended) Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B : LU , QR , SAS^{-1} , QAQ^T ?

A : Column vectors are independent and orthogonal.

$\therefore A$ is invertible and orthogonal

$A^T = A$, so A is diagonalizable

The sum of the column entries is 1 for each column. $\therefore A$ is a Markov matrix.

Since $AA^T = I$, then A is a permutation matrix, a permutation is orthogonal, so $P^{-1} = P^T$. $\therefore P^T P = I$.

Since A is invertible, $\lambda = 0$ is not an eigenvalue. $\therefore A$ cannot "project" vectors onto a subspace of the column vectors.

A invertible \Rightarrow can do $A = LU$
also, column vectors are independent so

can do Gram-Schmidt, so can do
 $A = QR$

A symmetric \Rightarrow can do $Q\Lambda Q^T$
which is a special case of $S\Lambda S^{-1}$

B: Column vectors dependent so not invertible,
not orthogonal.

$B^T B \neq I$, so B not a permutation

Since B is singular, $\lambda = 0$ is an
eigenvalue, so B can project vectors
onto a subspace of column vectors
 \therefore a projection matrix

Since $B = B^T$, B is diagonalizable

Since column entries add to 1, B is Markov

Can always do $B = LU$, although U
will not be invertible. (zero in a pivot).

Since columns of B are dependent, can't
do Gram-Schmidt. \therefore no $B = QR$

Since $B = B^T$, can do $B = Q\Lambda Q^T$,
which is a special case of $S\Lambda S^{-1}$