6.5 Positive Definite Matrices

Note Title 4/30/2008

1. Which of A_1, A_2, A_3, A_4 has two positive eigenvalues? Use the test, don't compute the λ's. Find an x so that $x^T A_1 x < 0$.

$$A_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

$\det A_1 = -1$, no

A_2 has $a_{11} < 0$, no

$\det A_3 = 0$, no

$\det A_4 = 1$, $a_{11} > 0$, so A_4 is pos. definite

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5x^2 + 12xy + 7y^2$$

$$= 5\left(x^2 + \frac{12}{5}xy\right) + 7y^2$$

$$= 5\left(x + \frac{12}{10}y\right)^2 - (5)\frac{36}{25}y^2 + 7y^2$$

$$= 5\left(x + \frac{6}{5}y\right)^2 + 7y^2 - \frac{36}{5}y^2$$

$$= 5\left(x + \frac{6}{5}y\right)^2 - \frac{1}{5}y^2$$

$$\therefore\ x^T A_1 x < 0 \iff 5\left(x + \frac{6}{5}y\right)^2 < \frac{1}{5}y^2$$

$$\iff \frac{1}{5}(5x + 6y)^2 < \frac{1}{5}y^2$$

$$\iff (5x + 6y)^2 < y^2$$

$\therefore$ if $5x + 6y = 0$, and $y \neq 0$, $x^T A_1 x < 0$

$\therefore$ $x = 6$, $y = -5$. $\therefore$ $\underline{x = (6, -5)}$

4. Show that $f(x, y) = x^2 + 4xy + 3y^2$ does not have a minimum at $(0, 0)$ even though it has positive coefficients. Write f as a *difference* of squares and find a point (x, y) where f is negative.

$$f(x,y) = [x\ y] \begin{bmatrix} 1 & 2 \\ 2 & \sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 2 \\ 2 & \sqrt{3} \end{bmatrix}$ is not positive definite

$$f(x,y) = x^2 + 4xy + 4y^2 - 4y^2 + 3y^2$$

$$= (x+2y)^2 - y^2$$

So, when $x = -2y$, $(x+2y)^2 = 0$, and $-y^2 < 0$ when $y \neq 0$.

$\therefore$ Let $y = -1$, $x = 2$

$\therefore$ $f(2, -1) < f(0, 0)$

5. The function $f(x, y) = 2xy$ certainly has a saddle point and not a minimum at $(0, 0)$. What symmetric matrix A produces this f? What are its eigenvalues?

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \therefore \lambda_1 + \lambda_2 = 0, \text{ and } \lambda_1 \lambda_2 = -1$$

$$\therefore \lambda_1 = 1, \ \lambda_2 = -1$$

6. (*Important*) If A has independent columns then $A^{\mathrm{T}}A$ is square and symmetric and invertible (Section 4.2). ***Rewrite*** $\boldsymbol{x^{\mathrm{T}}A^{\mathrm{T}}Ax}$ ***to show why it is positive except when*** $\boldsymbol{x = 0}$. Then $A^{\mathrm{T}}A$ is more than invertible, it is positive definite.

$$x^T A^T A x = (Ax)^T A x = \|Ax\|^2 \geq 0$$

$Ax = 0$ when $x = 0$. $A \neq 0$ since A is invertible.

7. Test to see if $A^{\mathrm{T}}A$ is positive definite in each case:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$$ det = 9, so pos. def.

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$$ det = 11, so pos. def.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

$$\det(A^TA) = 2(9) - 3(3) + 3(-3) = 0$$

$\therefore$ Some λ must be 0, so not pos. def.

Also,
$$\begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 3 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 3 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

8. The function $f(x, y) = 3(x+2y)^2 + 4y^2$ is positive except at $(0, 0)$. What is the matrix in $f = [x \;\; y]A[x \;\; y]^{\mathrm{T}}$? Check that the pivots of A are 3 and 4.

$$f(x,y) = 3x^2 + 12xy + 16y^2$$
$$= [x\ y]\begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 6 \\ 0 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
$$= LDL^T$$

Pivots are in D, and are 3 and 4.

Let $A = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} a & 0 \\ la & c \end{bmatrix} \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & la \\ la & l^2a+c \end{bmatrix}$

$\therefore [x\ y] A \begin{bmatrix} x \\ y \end{bmatrix} = [x\ y] \begin{bmatrix} a & la \\ la & l^2a+c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$[ax+lay \quad lax+l^2ay+cy] \begin{bmatrix} x \\ y \end{bmatrix}$

$= ax^2 + laxy + laxy + l^2ay^2 + cy^2$

$= a(x^2 + 2lxy + l^2y^2) + cy^2$

$= a(x+ly)^2 + cy^2$

9. Find the 3 by 3 matrix A and its pivots, rank, eigenvalues, and determinant:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

Let $A = L \Lambda L^T$

$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ a l_{21} & b & 0 \\ a l_{31} & b l_{32} & c \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$xL = \begin{bmatrix} x_1 + l_{21} x_2 + l_{31} x_3 & x_2 + l_{32} x_3 & x_3 \end{bmatrix}$$

$$xLD = \begin{bmatrix} x_1 + l_{21} x_2 + l_{31} x_3 & x_2 + l_{32} x_3 & x_3 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$= \begin{bmatrix} a x_1 + a l_{21} x_2 + a l_{31} x_3 & b x_2 + b l_{32} x_3 & c x_3 \end{bmatrix}$$

$$L^T x^T = (xL)^T = \begin{bmatrix} x_1 + l_{21} x_2 + l_{31} x_3 & x_2 + l_{32} x_3 & x_3 \end{bmatrix}^T$$

$$\begin{aligned} \therefore xLDL^Tx^T = {} & a x_1^2 + a l_{21} x_1 x_2 + a l_{31} x_1 x_3 \\ & + a l_{21} x_1 x_2 + a l_{21}^2 x_2^2 + a l_{21} l_{31} x_2 x_3 \\ & + a l_{31} x_1 x_3 + a l_{21} l_{31} x_2 x_3 + a l_{31}^2 x_3^2 \\ & + b x_2^2 + b l_{32} x_2 x_3 + b l_{32} x_2 x_3 + b l_{32}^2 x_3^2 \\ & + c x_3^2 \end{aligned}$$

$$= a x_1^2 + (a l_{21}^2 + b) x_2^2 + (a l_{31}^2 + b l_{32}^2 + c) x_3^2$$

$$+ 2a l_{21} x_1 x_2 + 2a l_{31} x_1 x_3 + (2a l_{21} l_{31} + 2b l_{32}) x_2 x_3$$

Note $LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} a & 0 & 0 \\ a l_{21} & b & 0 \\ a l_{31} & b l_{32} & c \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a l_{21} & a l_{31} \\ a l_{21} & a l_{21}^2 + b & a l_{21} l_{31} + b l_{32} \\ a l_{31} & a l_{21} l_{31} + b l_{32} & a l_{31}^2 + b l_{32}^2 + c \end{bmatrix}$$

<u>coefficients</u> $= \begin{bmatrix} x_1^2 & \frac{1}{2} x_1 x_2 & \frac{1}{2} x_1 x_3 \\ \frac{1}{2} x_1 x_2 & x_2^2 & \frac{1}{2} x_2 x_3 \\ \frac{1}{2} x_1 x_3 & \frac{1}{2} x_2 x_3 & x_3^2 \end{bmatrix}$

$$\therefore 4(x_1 - x_2 + 2x_3)^2 = 4(x_1 - x_2 + 2x_3)(x_1 - x_2 + 2x_3)$$

$$= 4(x_1^2 - x_1 x_2 + 2x_1 x_3 - x_1 x_2 + x_2^2 - 2x_2 x_3$$

$$+ 2x_1 x_3 - 2x_2 x_3 + 4x_3^2)$$

$$= 4x_1^2 + 4x_2^2 + 16x_3^2 - 8x_1 x_2 + 16 x_1 x_3 - 16 x_2 x_3$$

$\therefore$ $\underline{a = 4}$, $2a l_{21} x_1 x_2 = -8 x_1 x_2$, $\underline{l_{21} = -1}$

$a l_{21}^2 + b = 4$, $\underline{b = 0}$

$2al_{31} = 16, \quad \underline{l_{31} = 2}$

$al_{31}^2 + bl_{32}^2 + c = 16, \quad \underline{c = 0}$

l_{32} is undetermined.

$$\therefore LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 4 & 0 & 0 \\ -4 & 0 & 0 \\ 8 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} = \underline{\begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}}$$

rank = 1

determinant = $a \cdot b \cdot c = 0$

pivot = 4

$\lambda_1 \lambda_2 \lambda_3 = 0, \quad \lambda_1 + \lambda_2 + \lambda_3 = 24$

only one pivot $\Rightarrow$ only one eigenvalue,

$\therefore \lambda_1 = 24, \lambda_2 = 0, \lambda_3 = 0.$

10. Which 3 by 3 symmetric matrices A produce these functions $f = \mathbf{x}^T A \mathbf{x}$? Why is the first matrix positive definite but not the second one?

(a) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$

(b) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$.

(a) From #9, $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Upper left determinants: 2, 3, 2(3)-2
so all 3 are positive $\Rightarrow$ pos. def.

(b) from #9, $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Note that $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

so A is singular.

$\therefore$ det $A = 0 \Rightarrow$ not pos. def.

11. Compute the three upper left determinants to establish positive definiteness. Verify that their ratios give the second and third pivots.

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}.$$

1×1: $\underline{2}$

2×2: $10 - 4 = \underline{6}$

3×3: $2(40-9) - 2(16-0) = 62 - 32 = \underline{30}$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 8 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

$2 = 2$

$3 = 6/2$

$5 = 30/6$

14. *If A is positive definite then A^{-1} is positive definite.* Best proof: The eigenvalues of A^{-1} are positive because _____. *Second proof* (only for 2 by 2):

The entries of $A^{-1} = \frac{1}{ac-b^2}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$ pass the determinant tests _____.

(a) $Ax = \lambda x \Rightarrow A^{-1}Ax = A^{-1}\lambda x \Rightarrow x = \lambda A^{-1}x$

Since $\lambda \neq 0$, $A^{-1}x = \frac{1}{\lambda}x$.

Since $\frac{1}{\lambda} > 0$ when $\lambda > 0$, A^{-1} is pos. def.

(b) Since A is pos. def., $a > 0$ and $ac - b^2 > 0$.

$\therefore ac > b^2$, $c > \frac{b^2}{a} > 0$. $\therefore$ $\underline{\underline{c > 0}}$

$\det A^{-1} = \frac{1}{\det A} > 0$ since $\det A > 0$.