6.6 Similar Matrices

Note Title 5/30/2008

1. If $B = M^{-1}AM$ and also $C = N^{-1}BN$, what matrix T gives $C = T^{-1}AT$? Conclusion: If B is similar to A and C is similar to B, then _____.

$C = N^{-1}(M^{-1}AM)N = N^{-1}M^{-1}AMN$

$= (MN)^{-1}A(MN)$

$\therefore T = MN$

A and C are similar

2. If $C = F^{-1}AF$ and also $C = G^{-1}BG$, what matrix M gives $B = M^{-1}AM$? Conclusion: If C is similar to A and also to B then _____.

$F^{-1}AF = G^{-1}BG \Rightarrow GF^{-1}AFG^{-1} = B$

$\Rightarrow (FG^{-1})^{-1}A(FG^{-1}) = B$

$\therefore M = FG^{-1}$

A and B are similar.

3. Show that A and B are similar by finding M so that $B = M^{-1}AM$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}.$$

Look at $MB = AM$, let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(a) $AM = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = MB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a & b \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & a+b \\ 0 & c+d \end{bmatrix}$

$\therefore a = 0,\ b = b,\ b = c+d$. Note $ad \neq bc$ since $\det(M) = ad - bc \neq 0$.

$\therefore$ try $b = c = 1,\ d = 0$

$\therefore \underline{M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$, $\underline{M^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

(b) $\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}\underset{M}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \underset{M}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}\underset{B}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$

$\Rightarrow \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} a-b & b-a \\ c-d & d-c \end{bmatrix}$

$\Rightarrow c = -b,\ d = -a$, and $ad \neq bc$

$\therefore$ let $a = 1,\ d = -1,\ b = c = 0$

$\therefore \underline{M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$, $\underline{M^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$

(c) $\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{M}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \underset{M}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \underset{B}{\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}}$

$\Rightarrow \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} 4a+2b & 3a+b \\ 4c+2d & 3c+d \end{bmatrix}$

$\therefore 2c = 3a+2b \qquad 3a = 2d$

$3a = 2d \qquad 3b+3d = 3c \quad (b+d=c)$

$\therefore$ let $a=0, d=0 \quad \therefore b+0=c$

and $ad \neq bc$, or $0 \neq bc$

$\therefore$ let $b=c=1$

$\therefore M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

4. If a 2 by 2 matrix A has eigenvalues 0 and 1, why is it similar to $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$? Deduce from Problem 2 that all 2 by 2 matrices with those eigenvalues are similar.

(a) Because $A = S_A \Lambda S_A^{-1}$, or $\Lambda = S_A^{-1} A S_A$, where S_A is the eigenvector matrix.

(b) Let B be any other 2x2 matrix with

The eigenvalues of 0 and 1. $\therefore B = S_B \Lambda S_B^{-1}$, where S_B is the eigenvector matrix for B. $\therefore \Lambda = S_B^{-1} B S_B = S_A^{-1} A S_A$

$\therefore$ From (2) above, A and B are similar.

As in (2), $B = S_B S_A^{-1} A S_A S_B^{-1}$

$$= (S_A S_B^{-1})^{-1} A (S_A S_B^{-1})$$

15. Prove that $\det(A - \lambda I) = \det(M^{-1}AM - \lambda I)$. (You could write $I = M^{-1}M$ and factor out $\det M^{-1}$ and $\det M$.) This says that A and $M^{-1}AM$ have the same characteristic polynomial. So their roots are the same eigenvalues.

In general, $\det(X^{-1}AX) = \det(X^{-1})(\det A)(\det X)$
$= \det A$
since $\det(X^{-1})(\det X) = 1$.

$$\therefore \det(A - \lambda I) = \det(M^{-1})(\det(A - \lambda I))\det(M)$$
$$= \det[(M^{-1})(A - \lambda I)(M)]$$
$$= \det(M^{-1}AM - \lambda M^{-1}IM)$$
$$= \det(M^{-1}AM - \lambda I)$$

17. True or false, with a good reason:

(a) An invertible matrix can't be similar to a singular matrix.

(b) A symmetric matrix can't be similar to a nonsymmetric matrix.

(c) A can't be similar to $-A$ unless $A = 0$.

(d) A can't be similar to $A + I$.

(a) If A is invertible, so is $M^{-1}AM$

$\therefore$ <u>True</u>

(b) <u>False</u>. If A is nonsymmetric, and $n \times n$ with n different eigenvalues, Then $A = S\Lambda S^{-1}$, or $S^{-1}AS = \Lambda$.

$\therefore A$ is similar to symmetric Λ.

(c). <u>False</u>. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. $\therefore -A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\therefore \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} = \begin{bmatrix} b & -a \\ d & -c \end{bmatrix} \quad \therefore \begin{matrix} b = c \\ d = -a \end{matrix}$$

Let $b = c = 0$, $a = 1$, $d = -1$.

$\therefore M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $M^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\therefore M^{-1} A M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$$

(d) <u>True</u>. Eigenvalues of A and $A+I$ are offset by 1.

18. If B is invertible prove that AB *has the same eigenvalues as* BA.

Strategy: show AB and BA are similar.

Let $M = B$, $\therefore M^{-1} = B^{-1}$

$$\therefore M^{-1}(BA)M = B^{-1}(BA)(B)$$

$$= AB$$