7.1 The Idea of a Linear Transformation

Note Title 9/29/2008

1. A linear transformation must leave the zero vector fixed: $T(\mathbf{0}) = \mathbf{0}$. Prove this from $T(\boldsymbol{v}+\boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ by choosing $\boldsymbol{w} =$ _____. Prove it also from requirement (b) by choosing $c =$ _____.

Let $w = -v$.

Then
$$T(v+w) = T(v+(-v)) = T(0)$$
$$= T(v) + T(w)$$
$$= T(v) + T(-v)$$
$$= T(v) - T(v) = 0$$

Or, let $w=0$. Then
$$T(v+w) = T(v+0) = T(v)$$
$$= T(v) + T(0)$$
$$\therefore T(v) = T(v) + T(0) \Rightarrow T(0) = 0$$

Also, Let $c=0$. Then
$$T(cv) = T(0)$$
$$= cT(v) = 0$$

Or, let $c=-1$. Then
$$T(cv) = T(-v) = -T(v)$$
$$\Rightarrow T(v) + T(-v) = 0$$
$$= T(v+(-v)) = T(0)$$

4. If S and T are linear transformations, is $S(T(\boldsymbol{v}))$ linear or quadratic?

(a) (Special case) If $S(\boldsymbol{v}) = \boldsymbol{v}$ and $T(\boldsymbol{v}) = \boldsymbol{v}$, then $S(T(\boldsymbol{v})) = \boldsymbol{v}$ or $\boldsymbol{v}^2$?

(b) (General case) $S(\boldsymbol{w}_1+\boldsymbol{w}_2) = S(\boldsymbol{w}_1)+S(\boldsymbol{w}_2)$ and $T(\boldsymbol{v}_1+\boldsymbol{v}_2) = T(\boldsymbol{v}_1)+T(\boldsymbol{v}_2)$ combine into

$$S(T(\boldsymbol{v}_1+\boldsymbol{v}_2)) = S(\_\_\_\_) = \_\_\_\_ + \_\_\_\_.$$

(a) $S(T(v)) = S(v) = v$

(b) $S(T(v_1 + v_2)) = S(T(v_1) + T(v_2))$

$= S(T(v_1)) + S(T(v_2))$

12. Suppose a linear T transforms $(1,1)$ to $(2,2)$ and $(2,0)$ to $(0,0)$. Find $T(v)$ when

(a) $v = (2,2)$ (b) $v = (3,1)$ (c) $v = (-1,1)$ (d) $v = (a,b)$.

(a) $v = 2(1,1)$. $\therefore T((2,2)) = T(2(1,1)) = 2T((1,1)) = \underline{(4,4)}$

(b) $(3,1) = (1,1) + (2,0)$

$\therefore T(3,1) = T(1,1) + T(2,0) = (2,2) + (0,0) = \underline{(2,2)}$

(c) $(-1,1) = (1,1) - (2,0)$

$\therefore T(-1,1) = T(1,1) - T(2,0) = (2,2) - (0,0) = \underline{(2,2)}$

(d) $c(1,1) + d(2,0) = (a,b)$

$$\therefore \left.\begin{array}{l} c + 2d = a \\ c + 0 = b \end{array}\right\} c = b,\ d = \frac{a-b}{2}$$

$\therefore T((a,b)) = T(b(1,1)) + T(\frac{a-b}{2}(2,0))$

$= b\,T(1,1) + \frac{a-b}{2} T(2,0)$

$$= b(2,2) + \frac{a-b}{2}(0,0)$$

$$= \underline{(2b, 2b)}$$