7.3 Change of Basis

updated 10/31/2017

Note Title 10/7/2008

1. **1** Express the vectors $\boldsymbol{e} = (1,0,0,0)$ and $\boldsymbol{v} = (1,-1,1,-1)$ in the wavelet basis, as in equation (4). The coefficients c_1, c_2, c_3, c_4 solve $W\boldsymbol{c} = \boldsymbol{e}$ and $W\boldsymbol{c} = \boldsymbol{v}$.

From p. 386 of text,

$$W^{-1} = \begin{bmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/2 & \\ & & & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

(a) $e = (1,0,0,0)$

$$c = W^{-1}e, \therefore \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/2 & \\ & & & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/2 & \\ & & & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\therefore e = Wc = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$$

$$\therefore\ \underline{e = (1, 0, 0, 1/2)} = \underline{\tfrac{1}{4} w_1 + \tfrac{1}{4} w_2 + \tfrac{1}{2} w_3}$$

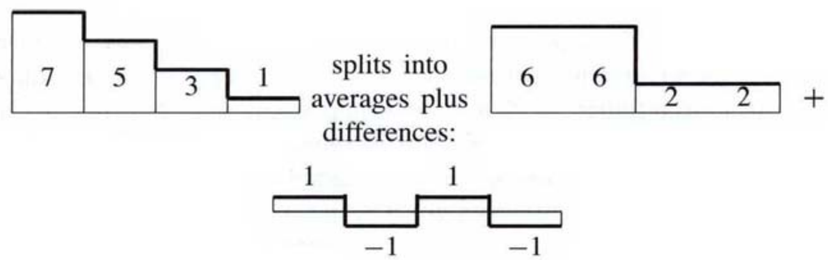
(5) $$c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/2 & \\ & & & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/2 & \\ & & & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$v = Wc = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\therefore\ \underline{v = (1, -1, 1, -1)} = \underline{w_3 + w_4}$$

2. **2** Follow Example 2 to represent $v = (7, 5, 3, 1)$ in the wavelet basis. Start with



The last step writes 6, 6, 2, 2 as an overall average plus a difference, using 1, 1, 1, 1 and 1, 1, −1, −1.

$$[7\ 5\ 3\ 1] = [6\ 6\ 2\ 2] + [1\ -1\ 1\ -1] \qquad c_3 = 1,\ c_4 = 1$$

$$[6\ 6\ 2\ 2] \downarrow [4\ 4\ 4\ 4] + [2\ 2\ -2\ -2] \qquad c_1 = 4,\ c_2 = 2$$

$$\begin{bmatrix}4\\4\\4\\4\end{bmatrix} = 4\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = 4w_1 \qquad \begin{bmatrix}2\\2\\-2\\-2\end{bmatrix} = 2\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} = 2w_2$$

$$\therefore c_1 = 4 \qquad \therefore c_2 = 2$$

$$\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = 1\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\\-1\end{bmatrix} = 1w_3 + 1w_4$$

$$\therefore [c_1, c_2, c_3, c_4] = \underline{[4, 2, 1, 1]}$$

3. **3** What are the eight vectors in the wavelet basis for $\mathbf{R}^8$? They include the long wavelet $(1, 1, 1, 1, -1, -1, -1, -1)$ and the short wavelet $(1, -1, 0, 0, 0, 0, 0, 0)$.

$w_1 = [1, 1, 1, 1, 1, 1, 1, 1]$

$w_2 = [1, 1, 1, 1, -1, -1, -1, -1]$

$w_3 = [1, 1, -1, -1, 0, 0, 0, 0]$

$w_4 = [1, -1, 0, 0, 0, 0, 0, 0]$

$w_5 = [0, 0, 1, -1, 0, 0, 0, 0]$

$w_6 = [0, 0, 0, 0, 1, 1, -1, -1]$

$w_7 = [0, 0, 0, 0, 1, -1, 0, 0]$

$w_8 = [0, 0, 0, 0, 0, 0, 1, -1]$

The process is to take a block (e.g., $[-1, -1, -1, -1]$) and derive sub-blocks using $1, -1$ to achieve orthogonality. $\therefore [-1, -1, -1, -1] \rightarrow [1, 1, -1, -1]$

$[1, -1, 0, 0]$

$[0, 0, 1, -1]$

4.

4 The wavelet basis matrix W factors into simpler matrices W_1 and W_2:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $W^{-1} = W_2^{-1} W_1^{-1}$ allows $\boldsymbol{c}$ to be computed in two steps. The first splitting in Example 2 shows $W_1^{-1}\boldsymbol{v}$. Then the second splitting applies W_2^{-1}. Find those inverse matrices W_1^{-1} and W_2^{-1} directly from W_1 and W_2. Apply them to $\boldsymbol{v} = (6, 4, 5, 1)$.

W_1: $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ subtracted row 1 from row 2, subtracted row 3 from row 4

$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$ added row 2 to row 4, added row 3 to row 2

$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$ subtracted row 2 from row 3

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

added row 3 to rows 2, 4

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{bmatrix}$$

multiplied row 3 by 1/2

multiplied row 4 by −1/2

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{bmatrix}$$

(−1) row 3 added to row 1

(−1) row 4 added to row 2

$$\therefore W_1^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{bmatrix}$$

Similarly, $W_2^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ using MATLAB

Perhaps a quicker way would be to observe colums of W_1 and W_2 are orthogonal, so $W_1^T W_1$ and $W_2^T W_2$ are diagonal.

$\therefore (W^T W)^{-1} = W^{-1}(W^T)^{-1}$, so $W^{-1} = (W^T W)^{-1} W^T$

$\therefore$ The inverse is just a (diagonal)(transpose)

$$W_1^{-1} v = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

$$W_2^{-1}(W_1^{-1} v) = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$ as in text, p. 386

5. **5** The 4 by 4 *Hadamard matrix* is like the wavelet matrix but entirely $+1$ and -1:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Find H^{-1} and write $\boldsymbol{v} = (7, 5, 3, 1)$ as a combination of the columns of H.

Note the columns are orthogonal. $\therefore$ $H^T H$ is a diagonal matrix. (Also note $H = H^T$).

$$\therefore (H^T H)^{-1} = H^{-1}(H^T)^{-1} \Rightarrow H^{-1} = (H^T H)^{-1} H^T$$

$$H^T H = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad \therefore (H^T H)^{-1} = \begin{bmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/4 & \\ & & & 1/4 \end{bmatrix}$$

$$\therefore H^{-1} = \begin{bmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/4 & \\ & & & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & -1/4 & 1/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & -1/4 & 1/4 \end{bmatrix}}} = \frac{1}{4} H$$

$$\therefore H^{-1}v = \frac{1}{4} H v = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 16 \\ 4 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$\therefore \underline{v = 4h_1 + h_2 + 2h_3}$$

6\. **6** Suppose we have two bases $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ for $\mathbf{R}^n$. If a vector has coefficients b_i in one basis and c_i in the other basis, what is the change of basis matrix in $\boldsymbol{b} = M\boldsymbol{c}$? Start from

$$b_1\boldsymbol{v}_1 + \cdots + b_n\boldsymbol{v}_n = V\boldsymbol{b} = c_1\boldsymbol{w}_1 + \cdots + c_n\boldsymbol{w}_n = W\boldsymbol{c}.$$

Your answer represents $T(\boldsymbol{v}) = \boldsymbol{v}$ with input basis of $\boldsymbol{v}$'s and output basis of $\boldsymbol{w}$'s. Because of different bases, the matrix is not I.

Since $Vb = Wc$, and V is invertible,

$b = V^{-1}Wc$, so $\underline{M = V^{-1}W}$

7. **7** The dual basis vectors $\boldsymbol{w}_1^*, \ldots, \boldsymbol{w}_n^*$ are the columns of $W^* = (W^{-1})^{\mathrm{T}}$. Show that the original basis $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ is "the dual of the dual." In other words, show that the $\boldsymbol{w}$'s are the rows of $(W^*)^{-1}$. **Hint:** Transpose the equation $WW^{-1} = I$.

The "dual of the dual" is the dual of W^*,

which are the rows of $(W^*)^{-1}$

$$(W^*)^{-1} = \left[(W^{-1})^T\right]^{-1} = \left[(W^{-1})^{-1}\right]^T \quad \text{using } (A^T)^{-1} = (A^{-1})^T$$

$$= [W]^T$$

$\therefore$ Rows of $(W^*)^{-1}$ are the rows of W^T,

which are the columns of W, which are w_i