1.1 Mathematical Induction

Note Title 4/21/2004

1. a. $1+2+3\ldots+n = \frac{n(n+1)}{2}$ for all $n \geq 1$.

$1 = \frac{1\cdot(1+1)}{2} = 1$

Suppose $1+2+\cdots+k = \frac{k(k+1)}{2}$

Then $1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2} + (k+1)$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

So, $k \Rightarrow k+1$

b. $1+3+5+\cdots+(2n-1) = n^2$ for all $n \geq 1$.

$1 = 1$

Suppose $1+3+\cdots+(2k-1) = k^2$

Then $1+3+\cdots+(2k-1)+(2k+1) = k^2+2k+1$

$$= (k+1)^2$$

So, $k \Rightarrow k+1$

c. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$, all $n \geq 1$

$k=1$: $1\cdot 2 = \frac{1(1+1)(1+2)}{3} = \frac{1\cdot 2\cdot 3}{3} = 2$

Suppose statement is true k. Then,

$1\cdot 2 + \cdots + k(k+1) + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$

$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}$

$= \frac{(k+1)[k(k+2) + 3k+6]}{3} = \frac{(k+1)[k^2+5k+6]}{3}$

$= \frac{(k+1)[(k+2)(k+3)]}{3}$ So, $k \Rightarrow k+1$

d. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$, all $n \geq 1$

$k=1$: $1^2 = \frac{1(2\cdot 1-1)(2\cdot 1+1)}{3} = \frac{1\cdot 1\cdot 3}{3} = 1$

$k \Rightarrow k+1$:

$1^2 + 3^2 + \cdots (2k-1)^2 + (2k+1)^2 =$

$$\frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= \frac{k(2k-1)(2k+1)}{3} + \frac{3(2k+1)^2}{3}$$

$$= \frac{(2k+1)\left[k(2k-1) + 3(2k+1)\right]}{3}$$

$$= \frac{(2k+1)\left[2k^2 - k + 6k + 3\right]}{3}$$

$$= \frac{(2k+1)\left[2k^2 + 5k + 3\right]}{3} = \frac{(2k+1)(2k+3)(k+1)}{3}$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}.$$ So true for $k+1$.

e. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$, all $n \geq 1$

$k=1$: $1^3 = \left[\frac{1 \cdot 2}{2}\right]^2 = 1^2 = 1$

$k \Rightarrow k+1$:

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3$$

$$= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^2(k+1)}{4}$$

$$= \frac{(k+1)^2\left[k^2+4k+4\right]}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2$$ So, true for $k+1$

2. If $r \neq 1$, Then $a + ar + \dots + ar^n = \frac{a(r^{n+1}-1)}{r-1}$, $n \geq 1$

For $k=1$: $a + ar^1 = a(1+r) = \frac{a(r^2-1)}{r-1} = a(r+1)$

$k \Rightarrow k+1$:

$$a + ar + \dots + ar^n + ar^{n+1} = \frac{a(r^{n+1}-1)}{r-1} + ar^{n+1}$$

$$= \frac{a(r^{n+1}-1)}{r-1} + \frac{ar^{n+1}(r-1)}{r-1}$$

$$= \frac{ar^{n+1} - a + ar^{n+2} - ar^{n+1}}{r-1}$$

$$= \frac{ar^{n+2} - a}{r-1} = \frac{a(r^{n+2}-1)}{r-1}$$

So, true for $k+1$

3. $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \dots + a + 1)$, for $n \geq 1$

For $k=1$: $a^1 - 1 = a-1 = (a-1)(a^0) = a-1$

$k \Rightarrow k+1$: $a^{k+1} - 1 = a^{k+1} - a^k - a + a^k + a - 1$

$$= a^{k+1} - a + a^k - 1 - a^k + a$$

$$= a(a^k - 1) + a^k - 1 - a(a^{k-1} - 1)$$

$$= (a+1)(a^k - 1) - a(a^{k-1} - 1)$$

Use 2nd principle of finite induction for $k, k-1$

$$= (a+1)\left[(a-1)(a^{k-1} + a^{k-2} + \dots + a + 1)\right]$$

$$- a\left[(a-1)(a^{k-2} + a^{k-3} + \dots + a + 1)\right]$$

$$= (a-1)\Big[a(a^{k-1} + a^{k-2} + \dots + a + 1)$$

$$\left.\begin{array}{l} + (a^{k-1} + a^{k-2} + \dots + a + 1) \\ - a(a^{k-2} + a^{k-3} + \dots + a + 1) \Big] \end{array}\right\} \text{combine these two}$$

$$= (a-1)\Big[a(a^{k-1}+a^{k-2}+\dots+a+1)$$
$$+ (a^{k-1}+a^{k-2}+\dots+a+1)$$
$$- (a^{k-1}+a^{k-2}+\dots+a^2+a)\Big]$$

$$= (a-1)\left[(a^k+a^{k-1}+\dots+a^2+a)+1\right]$$

and so, works for $k+1$

4. Cube of any integer can be written as the difference of two squares.

Proof: $n^3 = (1^3+2^3+\dots+n^3) - (1^3+2^3+\dots+(n-1)^3)$, all n

From 1(e), $1^3+2^3+\dots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$, $n \geq 1$

So, $n^3 = \left[\frac{n(n+1)}{2}\right]^2 - \left[\frac{(n-1)n}{2}\right]^2$

If n is even, then $\frac{n}{2}$ is an integer.
If n is odd, then $n+1$ and $n-1$ are even, so $\frac{n+1}{2}$ and $\frac{n-1}{2}$ are integers.

$\therefore$ n^3 is the difference between squares.

5. (a). For $n=4$, $n!+1 = 25 = 5^2$

$n=5$, $n!+1 = 121 = 11^2$

$n=7$, $n!+1 = 5041 = 71^2$

(b). False

$(3\cdot2)! = 720 \neq 3!2! = 6\cdot2 = 12$

$(2+3)! = 120 \neq 2!+3! = 2+6$

6. a. $n! > n^2$ for $n \geq 4$

Proof: $4! = 24 > 16 > 4^2$

Suppose $k! > k^2$, for $k > 4$

$(k+1)! = (k+1)\cdot k! > (k+1)\cdot k^2$

$= k^3 + k^2$

Since $k > 4$, then $k > 2$, so $k^2 > 2k$.

Since $k > 1$, then $k^3 > 2k$, so $k^3 \geq 2k+1$

So, $k^3 + k^2 \geq k^2 + 2k + 1 = (k+1)^2$

So $(k+1)! > k^3 + k^2 \geq (k+1)^2$

So $(k+1)! > (k+1)^2$

So, $k \Rightarrow k+1$

6.6. $n! > n^3$, $n \geq 6$

Proof: $6! = 720 > 216 = 6^3$

Suppose $k! > k^3$ for any $k > 6$

$\therefore (k+1)! = (k+1)k! > (k+1)k^3$

$= k^4 + k^3$

Since $k > 6$, Then $k > 3$, so $k^2 > 3k$

Also, $k > 1$, so $k^2 > k$, and $k^2 \geq k+1$

So $k^4 = k^2 \cdot k^2 > 3k(k+1) = 3k^2 + 3k$

$\therefore k^3 + k^4 > k^3 + 3k^2 + 3k$

$\therefore k^4 + k^3 \geq k^3 + 3k^2 + 3k + 1 = (k+1)^3$

$\therefore (k+1)! > k^4 + k^3 \geq (k+1)^3$

$(k+1)! > (k+1)^3$

So, $k \Rightarrow k+1$

7. $1(1!) + 2(2!) + \ldots + n(n!) = (n+1)! - 1, \quad n \geq 1$

$K=1: \quad 1(1!) = 1 = (1+1)! - 1 = 2! - 1 = 1$

$K \Rightarrow K+1$: Let $1(1!) + 2(2!) + \cdots + k(k!) = (k+1)! - 1$

Then $1(1!) + \cdots + (k+1)(k+1)!$

$$= (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)!\,[1 + k+1] - 1$$

$$= (k+1)!\,(k+2) - 1$$

$$= (k+2)! - 1$$ So, true for $k+1$

8. a. $2 \cdot 6 \cdot 10 \cdot \ldots \cdot (4n-2) = \dfrac{(2n)!}{n!}, \quad n \geq 1$

$K=1: \quad 2 = \dfrac{2!}{1!} = 2$

$K \Rightarrow K+1$: Let $2 \cdot 6 \cdot 10 \cdots (4k-2) = \dfrac{(2k)!}{k!}$

Then $2 \cdot 6 \cdot 10 \cdots (4k-2)(4k+2) =$

$$\frac{(2k)!}{k!}(4k+2) = \frac{(2k)!}{k!}(2k+1)2$$

$$= \frac{(2k)!}{k!}(2k+1)\,2\,\frac{(k+1)}{(k+1)}$$

$$= \frac{(2k)!\,(2k+1)(2k+2)}{k!\,(k+1)} = \frac{(2k+2)!}{(k+1)!}$$

So, true for $k+1$

6. $2^n (n!)^2 \le (2n)!$, $n \ge 1$

From (a), $(2n)! = 2 \cdot 6 \cdot 10 \cdots (4n-2)\,(n!)$

So problem reduces to:

$$2^n (n!)^2 \le 2 \cdot 6 \cdot 10 \cdots (4n-2)\,(n!)$$

Or, $2^n (n!) \le 2 \cdot 6 \cdot 10 \cdots (4n-2)$

For $k=1$: $2^1(1!) = 2 \le 2$

$k \Rightarrow k+1$: Let $2^k(k!) \le 2 \cdot 6 \cdot 10 \cdots (4k-2)$

$$2^{k+1}(k+1)! = 2^k (k!) \cdot 2 \cdot (k+1)$$

$$= 2^k (k!) (2k+2)$$

$$< 2^k (k!) (4k+2)$$

$$\leq 2 \cdot 6 \cdot 10 \cdots (4k-2) (4k+2)$$

$$= 2 \cdot 6 \cdot 10 \cdot \cdots (4(k+1) - 2)$$

So, true for $k+1$

9. If $1+a > 0$, Then $(1+a)^n \geq 1 + na$, $n \geq 1$

$k=1$: $1+a \geq 1+a$

$k \Rightarrow k+1$: Let $(1+a)^k \geq 1 + ka$

$$(1+a)^{k+1} = (1+a)^k (1+a)$$

$$\geq (1+ka)(1+a)$$

$$= 1 + ka + a + ka^2$$

$$\geq 1 + ka + a \quad (a^2 > 0,\ so\ ka^2 > 0)$$

$= 1 + (k+1)a$

$\therefore (1+a)^{k+1} \geq 1 + (k+1)a$

So, it's true for $k+1$

10. a. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$, $n \geq 1$

$k=1$: $\frac{1}{1^2} = 1 \leq 2 - \frac{1}{1} = 1$

$k \Rightarrow k+1$: Let $\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$

Then, $\frac{1}{1^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$

Since $k \geq 1$, Then $k^2+2k < k^2+2k+1$,

Or, $\frac{k^2+2k}{(k+1)^2} < 1$, or $\frac{k(k+2)}{(k+1)^2} < 1$

$\therefore \frac{(k+1)+1}{(k+1)^2} < \frac{1}{k} \Rightarrow \frac{1}{k+1} + \frac{1}{(k+1)^2} < \frac{1}{k}$

$\therefore -\frac{1}{k} + \frac{1}{(k+1)^2} < -\frac{1}{k+1}$

$$\therefore\ 2-\frac{1}{k}+\frac{1}{(k+1)^2}<2-\frac{1}{k+1}$$

$$\therefore\ \frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{(k+1)^2}<2-\frac{1}{k+1}$$

So, $k \Rightarrow k+1$

6. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\dots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$

$k=1$: $\frac{1}{2}=2-\frac{1+2}{2^1}=2-\frac{3}{2}=\frac{1}{2}$

$k \Rightarrow k+1$: Let $\frac{1}{2}+\frac{2}{2^2}+\dots+\frac{k}{2^k}=2-\frac{k+2}{2^k}$

Then, $\frac{1}{2}+\dots\frac{k}{2^k}+\frac{k+1}{2^{k+1}}=2-\frac{k+2}{2^k}+\frac{k+1}{2^{k+1}}$

$$=2-\frac{k+2}{2^k}\cdot\frac{2}{2}+\frac{k+1}{2^{k+1}}$$

$$=2+\frac{(k+1)-(2k+4)}{2^{k+1}}=2+\frac{-k-3}{2^{k+1}}$$

$$=2-\frac{(k+1)+2}{2^{k+1}}$$ So, $k \Rightarrow k+1$

11. $(2n)!/2^n n!$ is an integer, $n \geq 0$

$n=0$: $0! = 1$ by definition. So $\frac{0!}{2^0\, 0!} = \frac{1}{1 \cdot 1} = 1$

$k \Rightarrow k+1$: Suppose $\frac{(2k)!}{2^k\, k!}$ is an integer.

$$\frac{[2(k+1)]!}{2^{k+1}(k+1)!} = \frac{(2k)!\,(2k+1)(2k+2)}{2^k (k!)\, 2(k+1)}$$

$$= \frac{(2k)!}{2^k\, k!} \cdot (2k+1)\frac{(2k+2)}{2k+2}$$

$$= (\text{integer}) \cdot (2k+1) = \text{integer}$$

12. $T(21) = 32$
$T(T(21)) = 16$
$T(16) = 8$
$T(8) = 4$
$T(4) = 2$
$T(2) = 1$

$T(23) = 35$
$T(35) = 53$
$T(53) = 80$
$T(80) = 40$
$T(40) = 20$
$T(20) = 10$
$T(10) = 5$
$T(5) = 8$
$T(8) = 4$
$T(4) = 2$
$T(2) = 1$

13. $a_1 = 1 \quad a_2 = 2 \quad a_3 = 3$

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$, for $n \geq 4$

Prove: $a_n < 2^n$, $n \geq 1$

Proof: $1 < 2^1$, $2 < 2^2$, $3 < 2^3$

Let $k \geq 4$, and assume $a_K < 2^K$, $K = 4, \dots, k$

Then $a_{K+1} = a_k + a_{K-1} + a_{K-2}$

$$< 2^K + 2^{K-1} + 2^{K-2}$$

$$< 2^K + 2^{K-1} + 2^{K-1}$$

$$= 2^K + 2 \cdot 2^{K-1} = 2^K + 2^K$$

$$= 2 \cdot 2^k = 2^{K+1}$$

$\therefore \; a_{K+1} < 2^{K+1}$

14. $a_1 = 11$, $a_2 = 21$, $a_n = 3a_{n-1} - 2a_{n-2}$, $n \geq 3$

Prove: $a_n = 5 \cdot 2^n + 1$, $n \geq 1$

Proof: $a_1 = 5 \cdot 2 + 1 = 11$

$a_2 = 5 \cdot 4 + 1 = 21$

Suppose $a_k = 5 \cdot 2^k + 1$ for $3, 4, \ldots, k$

Then $a_{k+1} = 3a_k - 2a_{k-1}$

$$= 3(5 \cdot 2^k + 1) - 2(5 \cdot 2^{k-1} + 1)$$

$$= 15 \cdot 2^k + 3 - 5 \cdot 2^k - 2$$

$$= 10 \cdot 2^k + 1$$

$$= 5 \cdot 2 \cdot 2^k + 1$$

$$= 5 \cdot 2^{k+1} + 1$$

$\therefore$ if works for $3, 4, \ldots, k$, Then works for $k+1$.

$\therefore$ Works for all $k \geq 1$

1.2 The Binomial Theorem

Note Title 4/25/2004

1. a. Newton's identity

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r} \qquad n \geq k \geq r \geq 0$$

$$\frac{n!}{k!(n-k)!} \frac{k!}{r!(k-r!)} = \frac{n!}{r!} \cdot \frac{1}{(n-k)!(k-r!)}$$

$$= \frac{n!}{r!} \cdot \frac{(n-r)!}{(n-r)!} \cdot \frac{1}{(n-k)!(k-r)!}$$

$$= \frac{n!}{r!(n-r)!} \cdot \frac{(n-r)!}{(k-r)!\,(n-k)!}$$

$$= \binom{n}{r} \cdot \frac{(n-r)!}{(k-r)!\,(n-r-(k-r))!} = \binom{n}{r}\binom{n-r}{k-r}$$

b. $\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1} \qquad n \geq k \geq 1$

Without using part (a),

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n! \cdot (n-k+1)}{k(k-1)!\,(n-k+1)!}$$

$$= \frac{n!}{(k-1)!\,(n-k+1)!} \cdot \frac{(n-k+1)}{k} = \frac{(n-k+1)}{k}\binom{n}{k-1}$$

To use part (a), Let $r=1$

Then $\binom{n}{k}\binom{k}{1} = \binom{n}{1}\binom{n-1}{k-r}$ $\quad n \geq k \geq r \geq 0$

So, $\binom{n}{k}k = n\binom{n-1}{k-1}$

$$= n \frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$= \frac{n!}{(k-1)!\,(n-k+1)!} \cdot (n-k+1)$$

So, $\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$

2. If $2 \leq k \leq n-2$, and $n \geq 4$

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$$

Working from the right hand side,

$$\frac{(n-2)!}{(k-2)!\,(n-k)!} + \frac{2\,(n-2)!}{(k-1)!\,(n-k-1)!} + \frac{(n-2)!}{k!\,(n-k-2)!}$$

$$= \frac{k\cdot(k-1)\,(n-2)!}{k!\,(n-k)!} + \frac{2k\,(n-k)\,(n-2)!}{k!\,(n-k)!} + \frac{(n-k)(n-k-1)(n-2)!}{k!\,(n-k)!}$$

$$= \frac{(n-2)!\,[k^2-k+2kn-2k^2+n^2-nk-n-kn+k^2+k]}{k!\,(n-k)!}$$

$$= \frac{(n-2)!\,[n^2-n]}{k!\,(n-k)!} = \frac{n(n-1)(n-2)!}{k!\,(n-k)!} = \binom{n}{k}$$

$2 \leq k$ for $(k-2)!$ in denominator to work
$n-k-2 \geq 0$, or $n-2 \geq k \geq 2$, so $n \geq 4$ for
$(n-k-2)!$ in denominator to work.

3. a. From binomial Theorem, letting $a=b=1$,

$$(a+b)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k$$

So, $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

b. From binomial Theorem, let $a=1$, $b=-1$

$$0^n = 0 = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n}$$

c. $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$

In binomial Theorem, let $a=1$

Then $(1+b)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} b^k$

So, $n(1+b)^{n-1} = n\left[\binom{n-1}{0} + \binom{n-1}{1} b + \cdots + \binom{n-1}{n-1} b^{k-1}\right]$

Now let $b=1$. Then

$$n2^{n-1} = n\binom{n-1}{0} + n\binom{n-1}{1} + \cdots n\binom{n-1}{n-1}$$

$$= \sum_{k=0}^{n-1} n\binom{n-1}{k}$$

But $n\binom{n-1}{k} = \dfrac{n(n-1)!}{k!(n-k-1)!} = \dfrac{n!}{k!(n-(k+1))!} \cdot \dfrac{(k+1)}{(k+1)}$

$$= (k+1)\frac{n!}{(k+1)!\,(n-(k+1))!}$$

$$= (k+1)\binom{n}{k+1}$$

$$\therefore n2^{n-1} = \sum_{k=0}^{n-1} n\binom{n-1}{k} = \sum_{k=0}^{n-1}(k+1)\binom{n}{k+1}$$

$$= \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n}$$

d. $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \dots + 2^n\binom{n}{n} = 3^n$

In binomial Theorem, let $a=1$, $b=2$

$$(a+b)^n = 3^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}2 + \dots + \binom{n}{n}2^n$$

$$= \binom{n}{0} + 2\binom{n}{1} + \dots + 2^n\binom{n}{n}$$

e. $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\dots = 2^{n-1}$

$\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\dots = 2^{n-1}$

Proof: Add / Subtract results of (a) & (b)

If n is even, Then last term is positive

$$\begin{aligned} &\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n} = 2^n \\ +&\left[\binom{n}{0}-\binom{n}{1}+\dots+\binom{n}{n} = 0\right] \\ \hline &2\left[\binom{n}{0}+\binom{n}{2}+\dots+\binom{n}{n}\right] = 2^n \end{aligned}$$

If n is odd, last term is $-\binom{n}{n}$

$$\begin{aligned} &\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n} = 2^n \\ +&\left[\binom{n}{0}-\binom{n}{1}+\dots-\binom{n}{n} = 0\right] \\ \hline &2\left[\binom{n}{0}+\binom{n}{2}+\dots+\binom{n}{n-1}\right] = 2^n \end{aligned}$$

So, $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\dots = 2^{n-1}$

If n is even, Then last term is positive

$$\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n$$
$$-\left[\binom{n}{0}-\binom{n}{1}+\cdots+\binom{n}{n}=0\right]$$
$$2\left[\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n-1}\right]=2^n$$

If n is odd, last term is $-\binom{n}{n}$

$$\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n$$
$$-\left[\binom{n}{0}-\binom{n}{1}+\cdots-\binom{n}{n}=0\right]$$
$$2\left[\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n}\right]=2^n$$

So, $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots=2^{n-1}$

f. $\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\cdots+\frac{(-1)^n}{n+1}\binom{n}{n}=\frac{1}{n+1}$

The terms look like the terms in (6) with coefficients. So, need a relation with coefficient in front of binomial term.

The "k"th term can be written as:

$$(-1)^{k-1} \cdot \frac{1}{k}\binom{n}{k-1}$$

Note that $\binom{n}{k-1} = \dfrac{n!}{(k-1)!\,(n-k+1)!}$

$$= \frac{k}{n+1} \cdot \frac{(n+1)!}{k!\,(n-k+1)!}$$

Thus, $\dfrac{1}{k}\binom{n}{k-1} = \dfrac{1}{n+1}\binom{n+1}{k}$

So, problem is equivalent to:

$$\underset{(k=1)}{\binom{n}{0}} - \underset{(k=2)}{\frac{1}{2}\binom{n}{1}} + \cdots + \underset{(k=n+1)}{\frac{(-1)^n}{n+1}\binom{n}{n}}$$

$$= \frac{1}{n+1}\binom{n+1}{1} - \frac{1}{n+1}\binom{n+1}{2} + \cdots + \frac{(-1)^n}{n+1}\binom{n+1}{n+1}$$

$$= \frac{1}{n+1}\left[\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \cdots (-1)^n\binom{n+1}{n+1}\right]$$

From (6), $\binom{n}{0} = \binom{n}{1} - \binom{n}{2} + \cdots - (-1)^n \binom{n}{n}$

Substituting $n = s+1$,

$$\binom{s+1}{0} = \binom{s+1}{1} - \binom{s+1}{2} + \cdots - (-1)^{s+1} \binom{s+1}{s+1}$$

$$1 = \binom{s+1}{1} - \binom{s+1}{2} + \cdots + (-1)^s \binom{s+1}{s+1}$$

$$\therefore \binom{n}{0} - \frac{1}{2}\binom{n}{1} + \cdots + \frac{(-1)^n}{n+1}\binom{n+1}{n+1}$$

$$= \frac{1}{n+1}\left[\binom{n+1}{1} - \binom{n+1}{2} + \cdots + (-1)^n \binom{n+1}{n+1}\right]$$

$$= \frac{1}{n+1}\left[1\right] = \frac{1}{n+1}$$

4. a. For $n \geq 1$, $\binom{n}{r} < \binom{n}{r+1} \iff 0 \leq r < \frac{1}{2}(n-1)$

Proof: $\binom{n}{r} < \binom{n}{r+1}$

$$\iff \frac{n!}{r!(n-r)!} < \frac{n!}{(r+1)!(n-r-1)!}, \quad 0 \leq r, 0 \leq n-r-1$$

$$\iff \frac{(r+1)!}{r!} < \frac{(n-r)!}{(n-r-1)!}, \quad 0 \le r \le n-1$$

$$\iff r+1 < n-r, \quad 0 \le r \le n-1$$

$$\iff 0 \le 2r < n-1$$

$$\iff 0 \le r < \tfrac{1}{2}(n-1)$$

6. $\binom{n}{r} > \binom{n}{r+1} \iff n-1 \ge r > \frac{1}{2}(n-1)$

Proof: $\binom{n}{r} > \binom{n}{r+1}$

$$\iff \frac{n!}{r!(n-r)!} > \frac{n!}{(r+1)!(n-r-1)!}, \quad r \ge 0,\ n-r-1 \ge 0$$

$$\iff \frac{(r+1)!}{r!} > \frac{(n-r)!}{(n-r-1)!}, \quad r \ge 0,\ n-r-1 \ge 0$$

$$\iff r+1 > n-r, \quad n-1 \ge r \ge 0$$

$$\iff 2r > n-1, \quad n-1 \ge r \ge 0$$

$$\iff n-1 \ge r > \tfrac{1}{2}(n-1) \ge 0$$

c. $\binom{n}{r} = \binom{n}{r+1} \iff r = \frac{1}{2}(n-1)$

Proof: From The steps in (a) + (b),

$\binom{n}{r} = \binom{n}{r+1} \iff r+1 = n-r, \quad n-1 \geq r \geq 0$

$\iff 2r = n-1, \quad n-1 \geq r \geq 0.$

$\iff r = \frac{1}{2}(n-1), \quad n-1 \geq r \geq 0$

5. a. For $n \geq 2$, $\binom{2}{2} + \binom{3}{2} + \dots + \binom{n}{2} = \binom{n+1}{3}$

Proof: For $k=2$, $\binom{2}{2} = 1 = \binom{2+1}{3} = 1$

$k \Rightarrow k+1$: Assume $\binom{2}{2} + \dots + \binom{k}{2} = \binom{k+1}{3}$

Then, $\binom{2}{2} + \dots + \binom{k}{2} + \binom{k+1}{2}$

$= \binom{k+1}{3} + \binom{k+1}{2}$

$= \binom{k+2}{3}$ From Pascal's identity $\binom{r}{s} + \binom{r}{s-1} = \binom{r+1}{s}, \ 1 \leq s \leq r$

6. First, $m^2 = 2\binom{m}{2} + m$, $m \geq 2$

$$2\binom{m}{2} + m \iff 2\frac{m!}{2!(m-2)!} + m \quad , m \geq 2$$

$$\iff m(m-1) + m = m^2, \quad m \geq 2$$

Now, $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Proof: $1^2 + 2^2 + \cdots + n^2$

$$= 1 + 2\binom{2}{2} + 2 + 2\binom{3}{2} + 3 + \cdots + 2\binom{n}{2} + n$$

$$= (1 + 2 + \cdots + n) + 2\left[\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n}{2}\right]$$

$$= (1 + 2 + \cdots + n) + 2\binom{n+1}{3}$$

$$= (1 + 2 + \cdots + n) + 2\frac{(n+1)!}{3 \cdot 2 \cdot (n-2)!}$$

$$= \frac{n(n+1)}{2} + \frac{(n+1)(n)(n-1)}{3}$$

$$= \frac{3n(n+1) + 2(n+1)(n)(n-1)}{6}$$

$$= \frac{n(n+1)[3 + 2n - 2]}{6} = \frac{n(n+1)(2n+1)}{6}$$

c. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

Proof: From (b), $m^2 = 2\binom{m}{2} + m$, or

$$m(m-1) = 2\binom{m}{2}, \quad m \geq 2$$

$\therefore\ 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1)$

$= 2\binom{2}{2} + 2\binom{3}{2} + 2\binom{4}{2} + \cdots + 2\binom{n+1}{2}$

$= 2\left[\binom{n+2}{3}\right]$, from (a)

$= 2\,\dfrac{(n+2)!}{3!\,(n-1)!} = \dfrac{2\,(n+2)(n+1)(n)}{3\cdot 2\cdot 1}$

$= \dfrac{n(n+1)(n+2)}{3}$

6. $\binom{2}{2} + \binom{4}{2} + \cdots + \binom{2n}{2} = \dfrac{n(n+1)(4n-1)}{6}, \ n \geq 2$

Proof: First, $\binom{2m}{2} = \dfrac{(2m)!}{2!(2m-2)!} = \dfrac{2m(2m-1)}{2}$

$= 2m^2 - m = m^2 + m^2 - m$

$$= m^2 + 2\binom{m}{2}, \quad m \geq 2, \text{ from } 5(c)$$

$$\therefore \binom{2}{2} + \binom{4}{2} + \dots + \binom{2n}{2}$$

$$= 1 + \left[2^2 + 2\binom{2}{2} + \dots + n^2 + 2\binom{n}{2}\right]$$

$$= (1^2 + 2^2 + \dots + n^2) + 2\left[\binom{2}{2} + \dots + \binom{n}{2}\right]$$

$$= \frac{n(n+1)(2n+1)}{6} + 2\binom{n+1}{3}, \quad \text{from } 5(b), 5(a) \quad n \geq 2$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{2(n+1)!}{3 \cdot 2 \cdot (n-2)!}$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{2(n+1)(n)(n-1)}{6}$$

$$= \frac{n(n+1)[2n+1 + 2n-2]}{6}$$

$$= \frac{n(n+1)(4n-1)}{6}$$

7. For $n \geq 1$, $1^2 + 3^2 + \dots + (2n-1)^2 = \binom{2n+1}{3}$

Proof: $k=1$: $1^2 = 1 = \binom{2\cdot 1+1}{3} = \binom{3}{3} = 1$

$k \Rightarrow k+1$: Assume $1^2 + 3^2 + \dots (2k-1)^2 = \binom{2k+1}{3}$

Then, $1^2 + 3^2 + \dots + (2k-1)^2 + (2(k+1)-1)^2$

$$= 1^2 + \dots + (2k-1)^2 + (2k+1)^2$$

$$= \binom{2k+1}{3} + (2k+1)^2$$

$$= \frac{(2k+1)!}{3!(2k-2)!} + (2k+1)^2$$

$$= \frac{(2k+1)!(2k)(2k-1)}{3!(2k-2)!(2k)(2k-1)} + \frac{6\cdot(2k+1)^2\cdot(2k)!}{6\cdot(2k)!}$$

$$= \frac{(2k+1)!\left[(2k)(2k-1) + 6(2k+1)\right]}{3!(2k)!}$$

$$= \frac{(2k+1)!\left[4k^2 - 2k + 12k + 6\right]}{3!(2k)!}$$

$$= \frac{(2k+1)!\,[(2k+2)(2k+3)]}{3!\,(2k)!}$$

$$= \frac{(2k+3)!}{3!\,(2k+3-3)!} = \binom{2k+3}{3}$$

So, $k \Rightarrow k+1$

8. For $n \geq 1$, $\binom{2n}{n} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\, 2^{2n}$

$k=1$: $\binom{2}{1} = \frac{2!}{1!\,1!} = 2$, $\frac{1}{2} 2^2 = \frac{4}{2} = 2$

$k \Rightarrow k+1$: Suppose $\binom{2k}{k} = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\, 2^{2k}$

Then $\binom{2k+2}{k+1} = \frac{(2k+2)!}{(k+1)!\,(k+1)!}$

$$= \frac{(2k+2)(2k+1)\,(2k)!}{(k+1)(k+1)\,k!\,k!}$$

$$= \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \binom{2k}{k}$$

$$= \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \cdot \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k} 2^{2k}$$

$$= \frac{2(k+1)}{(k+1)(k+1)} \cdot \frac{1\cdot 3\cdot 5\cdots(2k-1)(2k+1)}{2\cdot 4\cdot 6\cdots 2k} 2^{2k}$$

$$= \frac{2}{(k+1)} \cdot \frac{1\cdot 3\cdot 5\cdots(2k+1)}{2\cdot 4\cdot 6\cdots 2k} 2^{2k}$$

$$= \frac{4}{(2k+2)} \cdot \frac{1\cdot 3\cdot 5\cdots(2k+1)}{2\cdot 4\cdot 6\cdots 2k} 2^{2k}$$

$$= \frac{1\cdot 3\cdot 5\cdots(2k+1)}{2\cdot 4\cdot 6\cdots(2k)(2k+2)} \cdot 2^{2k+2}$$

So, $k \Rightarrow k+1$

9. $2^n < \binom{2n}{n} < 2^{2n}$, for $n > 1$

Proof: $n! < 1\cdot 3\cdot 5\cdots(2n-1)$, for $n > 1$
since it's true for $k=2$ $(2 < 1\cdot 3)$
and if $k! < 1\cdot 3\cdots(2k-1)$, then
$(k+1)! = k!(k+1) < 1\cdot 3\cdot 5\cdots(2k-1)(k+1)$
$< 1\cdot 3\cdot 5\cdots(2k-1)(2k+1)$

Also, $2^n n! = 2\cdot 4\cdot 6 \cdots 2n$, since it's true for $k=1$, and $2^{k+1}(k+1)! =$
$2(k+1)2^k k! = 2(k+1)\cdot 2\cdot 4\cdot 6 \cdots 2k$
$= 2\cdot 4\cdot 6 \cdots 2k\cdot 2(k+1)$

So, $2^n n! < 1\cdot 3\cdot 5 \cdots (2n-1) 2^n$

$$\Rightarrow 1 < \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n n!} 2^n$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4 \cdots 2n} 2^n$$

$$\therefore 2^n < \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} 2^{2n} = \binom{2n}{n}, \text{ by } (8)$$

Now, since $2k-1 < 2k$ for $k \geq 1$,

Then $1\cdot 3\cdot 5 \cdots (2n-1) < 2\cdot 4\cdot 6 \cdots 2n$,

So, $\dfrac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} < 1$, for $n \geq 1$

$$\therefore \text{ by } (8), \binom{2n}{n} = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} 2^{2n} < 2^{2n}, \; n \geq 1$$

10. Given $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$, $n \geq 0$

Prove: $C_n = \frac{2(2n-1)}{n+1} C_{n-1}$, $n \geq 1$

Proof: $k=1$: $C_1 = \frac{2!}{1!\,2!} = 1$, $C_0 = \frac{0!}{0!\,1!} = 1$

$$\therefore \frac{2(2-1)}{1+1} = 1,$$

$$\text{So, } C_1 = \frac{2(2-1)}{1+1} C_0$$

$k \Rightarrow k+1$: Suppose $C_k = \frac{2(2k-1)}{k+1} C_{k-1}$

Then $C_{k+1} = \frac{(2k+2)!}{(k+1)!\,(k+2)!}$

$$= \frac{(2k+2)(2k+1)}{(k+1)(k+2)} \cdot \frac{(2k)!}{k!\,(k+1)!}$$

$$= \frac{2(2k+1)}{(k+2)} \cdot C_k = \frac{2(2k+1)}{(k+2)} \cdot \frac{2(2k-1)}{(k+1)} C_{k-1}$$

$$= \frac{2(2k+1)\cdot 2(2k-1)}{(k+2)(k+1)} \cdot \frac{(2k-2)!}{(k-1)!\,k!}$$

$$= \frac{2(2k+1)\cdot 2}{(k+2)} \cdot \frac{(2k-1)!}{(k-1)!\,(k+1)!}$$

$$= \frac{2(2k+1)}{(k+2)} \cdot \frac{2k}{k} \cdot \frac{(2k-1)!}{(k-1)!\,(k+1)!}$$

$$= \frac{2(2k+1)}{(k+2)} \cdot \frac{(2k)!}{k!\,(k+1)!}$$

$$= \frac{2[2(k+1)-1]}{[(k+1)+1]} C_{(k+1)-1}$$

So, $k \Rightarrow k+1$

1.3 Early Number Theory

Note Title 5/10/2004

1\. a. A number is triangular $\iff$ it is of the form $\frac{n(n+1)}{2}$ for some $n \geq 1$.

Proof: $1+2+3+\dots+n = \frac{n(n+1)}{2}$

from problem 1(a) of Problem Set 1.1.

So, if a number X is triangular then for some n, $1+2+\cdots+n = X$ by definition, and so $X = \frac{n(n+1)}{2}$

If $X = \frac{n(n+1)}{2}$ for some n, then

$X = 1+2+\cdots+n$

b. An integer n is triangular $\iff$ $8n+1$ is a perfect square.

Proof: If n is triangular, then there is a K such that $n = \frac{K(K+1)}{2}$

$\therefore 8n = 4K(K+1)$,

$$8n+1 = 4k(k+1)+1$$

$$= 4k^2+4k+1$$

$$= (2k+1)^2$$

$\therefore$ n triangular $\Rightarrow$ $8n+1$ is a perfect square

If $8n+1$ is a perfect square, then there is an integer k such that $k^2 = 8n+1$.

Note that $8n+1$ must be odd.

$\therefore$ k^2 is odd, and so k is odd.

$\therefore$ There is an s such that $2s+1 = k$

$$\therefore (2s+1)^2 = 8n+1$$

$$\therefore 4s^2+4s+1 = 8n+1$$

$$\therefore 4s(s+1) = 8n$$

$$\therefore \frac{s(s+1)}{2} = n$$

$\therefore$ $8n+1$ a perfect square $\Rightarrow$ n triangular

c. If a and b are consecutive triangular numbers, then $a+b$ is a perfect square.

Proof: Let $1+2+\cdots+n = a$

Then $1+2+\cdots+n+n+1 = b$

$$\therefore a+b = \frac{n(n+1)}{2} + \frac{n(n+1)}{2} + (n+1)$$

$$= n(n+1) + (n+1)$$

$$= (n+1)(n+1)$$

So, $a+b$ is a perfect square.

d. If n is triangular, then so are

$9n+1$, $25n+3$, and $49+6$.

Proof: Let $1+2+\cdots+k = n$

Then $9n+1 = 9\frac{k(k+1)}{2} + 1 = \frac{9k^2+9k+2}{2}$

$$= \frac{(3k+1)(3k+2)}{2} = \frac{s(s+1)}{2}$$

for $s = 3k+1$, and so by 1(a), $9n+1$ is triangular.

$$25n+3 = \frac{25k(k+1)}{2} + 3 = \frac{25k^2+25k+6}{2}$$

$$= \frac{(5k+2)(5k+3)}{2} = \frac{s(s+1)}{2}$$

for $s = 5k+2$, and so by 1(a), $25n+3$ is triangular.

$$49n+6 = \frac{49k(k+1)}{2} + 6 = \frac{49k^2+49k+12}{2}$$

$$= \frac{(7k+3)(7k+4)}{2} = \frac{s(s+1)}{2}$$

For $s = 7k+2$, and so by 1(a), $49+6$ is triangular.

2. $t_n = \binom{n+1}{2}$, $n \geq 1$, t_n the n^{th} triangular.

$$\binom{n+1}{2} = \frac{(n+1)!}{2!\,(n-1)!} = \frac{(n+1)n}{2}, \text{ so by } 1(a),$$

$$t_n = \binom{n+1}{2}$$

3. $t_1 + t_2 + \dots + t_n = \dfrac{n(n+1)(n+2)}{6}$, $n \geq 1$

Proof: From problem 1(c) of problem set 1.1,

$$1\cdot 2 + 2\cdot 3 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}, \; n \geq 1$$

$$\therefore \frac{1\cdot 2}{2} + \frac{2\cdot 3}{2} + \dots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

Note that each term k can be written as $\dfrac{k(k+1)}{2} = t_k$

$$\therefore t_1 + t_2 + \dots + t_n = \frac{n(n+1)(n+2)}{6}$$

The hint given: $t_{k-1} + t_k = k^2$

$$t_{k-1} = \frac{(k-1)(k-1+1)}{2} = \frac{k(k-1)}{2}$$

$$\therefore t_{k-1} + t_k = \frac{k(k-1)}{2} + \frac{k(k+1)}{2}$$

$$= \frac{k^2 - k + k^2 + k}{2} = k^2$$

You could prove the statement using

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ from}$$

5(b) of problem set 1.2, and breaking problem up into even + odd number of terms.

4. $9(2n+1)^2 = t_{9n+4} - t_{3n+1}$

Proof: Since $t_k = \dfrac{k(k+1)}{2}$,

$$t_{9n+4} = \frac{(9n+4)(9n+5)}{2}, \quad t_{3n+1} = \frac{(3n+1)(3n+2)}{2}$$

$$\therefore t_{9n+4} - t_{3n+1} = \frac{(81n^2+81n+20)-(9n^2+9n+2)}{2}$$

$$= \frac{72n^2+72n+18}{2} = 36n^2+36n+9$$

$$= 9(4n^2+4n+1) = 9(2n+1)^2$$

5. a. Find two triangular numbers, t_r and t_s, such that $t_r + t_s$ and $t_r - t_s$ are triangular.

The triangular numbers are:

$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$

$15 + 21 = 36, \quad 21 - 15 = 6$

6. Three successive triangular numbers whose product is a perfect square.

$$\frac{n(n+1)}{2} \cdot \frac{(n+1)(n+2)}{2} \cdot \frac{(n+2)(n+3)}{2} = k^2$$

$$(n+1)^2(n+2)^2 \cdot \frac{n \cdot (n+3)}{2} = 4k^2 = (2k)^2$$

So, if can find an n such that

$\frac{n(n+3)}{2}$ is a perfect square, problem would be solved. The perfect squares are $1, 4, 9, 16, 25, 36, 49, \ldots$

By trial and error, if $n = 3$, Then $\frac{3(3+3)}{2} = 9$

So, $t_3 \cdot t_4 \cdot t_5 = 6 \cdot 10 \cdot 15 = 900 = 30^2$

c. Three successive triangular numbers whose sum is a perfect square.

Trial & error works faster than trying to figure this out.

$t_5 + T_6 + t_7 = 15 + 21 + 28 = 64 = 8^2$

6.a. If t_n is a perfect square, then $t_{4n(n+1)}$ is also a perfect square.

Proof: Assume $t_n = K^2 = \frac{n(n+1)}{2}$

Then $2K^2 = n(n+1)$

$$t_{4n(n+1)} = \frac{4n(n+1)\left(4n(n+1)+1\right)}{2}$$

$$= \frac{4 \cdot 2k^2 \cdot \left[4n^2+4n+1\right]}{2}$$

$$= 4k^2(2n+1)^2$$

$$= \left[2k(2n+1)\right]^2$$, and so is a square.

6. $t_1 = 1$ is a perfect square

$t_{4\cdot 1(1+1)} = t_8 = 36$ is a perfect square

$t_{4\cdot 8(8+1)} = t_{288} = \frac{288(288+1)}{2} = 41,616 = 204^2$

7. $t_{n+1}^2 - t_n^2 = k^3$, for some integer k

Proof: $t_{n+1} = \frac{(n+1)(n+2)}{2}$, $t_n = \frac{n(n+1)}{2}$

$$\therefore t_{n+1}^2 - t_n^2 = \frac{(n+1)^2(n+2)^2 - (n+1)^2 n^2}{4}$$

$$= \frac{(n+1)^2\left[n^2+4n+4-n^2\right]}{4}$$

$$= \frac{(n+1)^2\cdot(4n+4)}{4} = (n+1)^3, \text{ for } n \geq 1$$

8. $$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\dots+\frac{1}{t_n} < 2$$

Proof: Each term can be written as

$$\frac{1}{\frac{k(k+1)}{2}} = \frac{2}{k(k+1)} = 2\left[\frac{1}{k}-\frac{1}{k+1}\right]$$

$$\therefore \frac{1}{1}+\frac{1}{3}+\dots+\frac{1}{t_n} = 2\left[\frac{1}{1}-\frac{1}{2}\right]+2\left[\frac{1}{2}-\frac{1}{3}\right]+\dots+2\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

$$= 2\left[\frac{1}{1}-\frac{1}{n+1}\right] = 2\left(1-\frac{1}{n+1}\right)$$

Since $n>0$, Then $n+1>0$, so $\frac{1}{n+1}>0$,

and $-\frac{1}{n+1}<0$, so $1-\frac{1}{n+1}<1$,

so $2\left(1-\frac{1}{n+1}\right)<2$.

$$\therefore \frac{1}{1}+\frac{1}{3}+\dots+\frac{1}{t_n} < 2$$

9. a. $t_x = t_y + t_z$, $x = \frac{n(n+3)}{2} + 1$, $y = n+1$, $z = \frac{n(n+3)}{2}$

Proof: $t_y + t_z = \frac{(n+1)(n+2)}{2} + \frac{\frac{n(n+3)}{2}\left[\frac{n(n+3)}{2}+1\right]}{2}$

$$= \frac{2\,\frac{(n+1)(n+2)}{2} + \frac{n(n+3)}{2}\left[\frac{n(n+3)}{2}+1\right]}{2}$$

$$= \frac{2\,\frac{[n^2+3n+2]}{2} + \frac{n(n+3)}{2}\left[\frac{n(n+3)}{2}+1\right]}{2}$$

$$= \frac{2\left[\frac{n(n+3)}{2}+1\right] + \frac{n(n+3)}{2}\left[\frac{n(n+3)}{2}+1\right]}{2}$$

$$= \frac{\left[\frac{n(n+3)}{2}+1\right]\left[\frac{n(n+3)}{2}+1+1\right]}{2}$$

$= t_x$, for $n \geq 1$ (if $n=0$, then $z=0$)

b. $n=1$: $t_3 = t_2 + t_2$, or $6 = 3+3$

$n=2$: $t_6 = t_3 + t_5$, or $21 = 6 + 15$

$n = 3$; $t_{10} = t_4 + t_9$, or $55 = 10 + 45$